Antipodal monochromatic paths in hypercubes

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Hypercubes

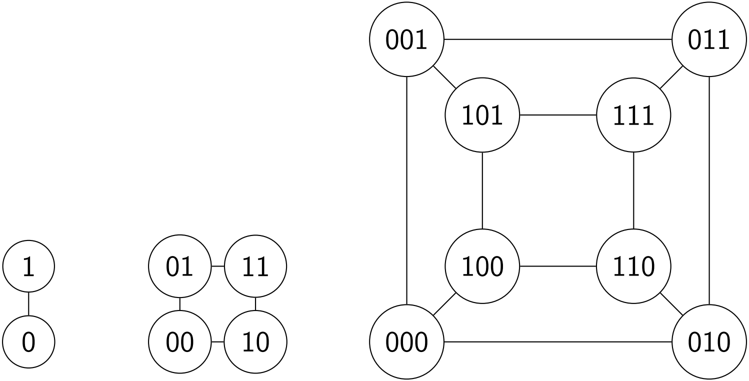


Figure: From left to right, graphs Q_1, Q_2 and Q_3.

Definition

The n-dimensional hypercube Q_n is an undirected graph with $V(Q_n) = \{0,1\}^n$ and $E(Q_n) = \{(u,v) : u \text{ and } v \text{ differ in exactly one coordinate}\}$.

Antipodal vertices

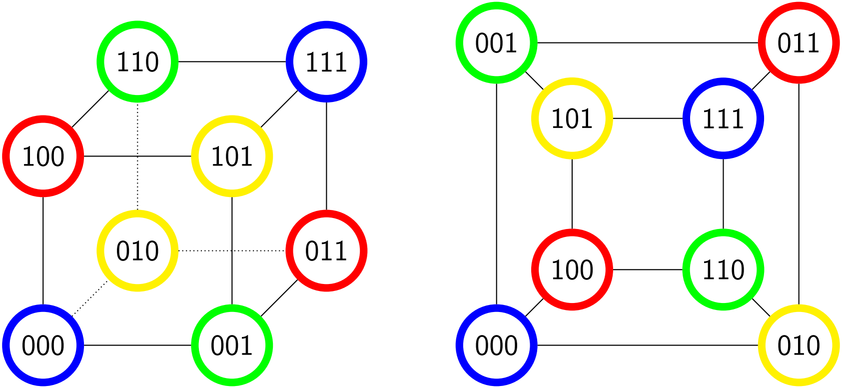


Figure: Q_3, antipodal vertices are marked with the same color.

Definition

Let u be a vertex of the hypercube Q_n. Its *antipodal vertex* u' is the vertex which differs from u in every coordinate.

Antipodal edges

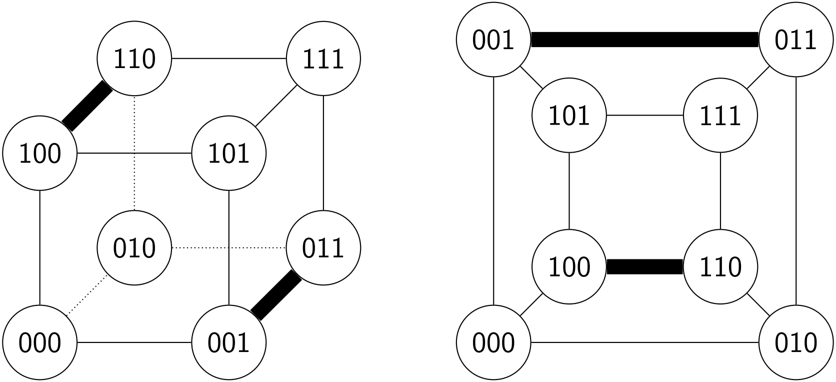


Figure: An example of a pair of antipodal edges in Q_3 is drawn by a thick line.

Definition

Let $e = (u, v)$ be an edge of the hypercube Q_n. Its *antipodal edge* is the edge $e' = (u', v')$.

Colorings

Definition

An *edge 2-coloring* is any mapping $c : E(Q_n) \to \{\text{red, blue}\}$.

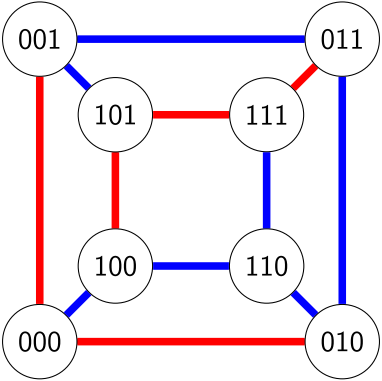
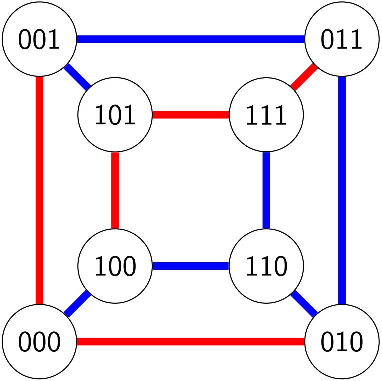


Figure: A 2-coloring of Q_3.

Question

Given *any* edge 2-coloring of a Q_n, is there always a pair of antipodal vertices such that there is a monochromatic path connecting them?



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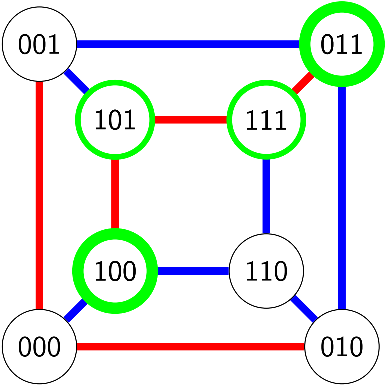

Figure: A 2-coloring of Q_3.

Not really :(

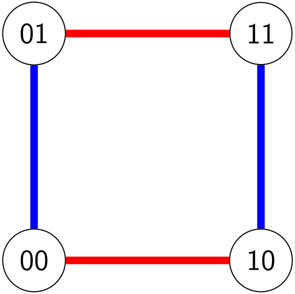

Figure: A possible coloring of Q_2.

Thank you for your attention!

Antipodal colorings

Definition

An edge 2-coloring is antipodal if all pairs of antipodal edges have different colors.

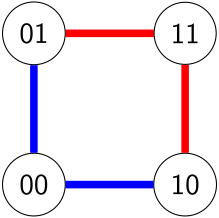

Figure: The only antipodal coloring of Q_2.

Conjecture (S. Norine)

For any antipodal coloring of a hypercube Q_n there always exists a pair of antipodal vertices $x, x' \in V(Q_n)$ such that there is a monochromatic path connecting x and x'.

Note

This conjecture has been verified for $n \leq 6$ (see [WW19]).

Loosening antipodality

Definition

A path P is a *k-switch* path for some $k \geq 0$ if P is a concatenation of at most $k+1$ monochromatic paths. Note that the coloring does not have to be antipodal.

Norine's conjecture restated: Is there always a 0-switch path between some pair of antipodal vertices of Q_n for all antipodal colorings?

Loosening antipodality

Conjecture (T. Feder, C. Subi) [FS13]

For any coloring[1] of a hypercube Q_n there always exists a pair of antipodal vertices $x, x' \in V(Q_n)$ such that there is a 1-switch path connecting x and x'.

Note

This conjecture has been verified for $n \leq 5$ in [FS13] and if it holds, it implies Norine's conjecture.

[1] Not necessarily antipodal.

Possible approaches

- Find an upper bound on the number of switches.

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- Generalize the conjecture to more general graphs than hypercubes (see [Sol17]).
- Determine the expected number of switches over all pairs of antipodal vertices in Q_n for fixed n.
- Fix a pair of antipodal vertices x, x' in Q_n. Determine the average number of switches between x and x' all possible colorings.

References

Vojtěch Dvořák.
A note on Norine's antipodal-colouring conjecture.
arXiv preprint arXiv:1912.07504, 2019.

Tomás Feder and Carlos Subi.
On hypercube labellings and antipodal monochromatic paths.
Discrete Applied Mathematics, 161(10-11):1421–1426, 2013.

Daniel Soltész.
On the 1-switch conjecture.
Discrete Mathematics, 340(7):1749–1756, 2017.

Douglas B West and Jennifer I Wise.
Antipodal edge-colorings of hypercubes.
Discussiones Mathematicae Graph Theory, 39(1):271–284, 2019.