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DIMACS REU 2020 Starting presentation

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Our project: Updating colorings

- We start with a (non-proper) coloring c_0 of a graph G using three (or more) colors, and a given sequence $v_1 \dots, v_T$ of vertices
- At time $k \in \{1, \dots T\}$, we update the color of v_k - we look at its neighborhood, and if it has x blue neighbors, the probability of the new color of v_k being blue is proportional to λ^x, where $\lambda > 1$ is some constant.

Problem:

Which choice of the initial coloring c_0 makes the event that at time T, all vertices are colored blue, most likely?

- The obvious answer seems to be color everything blue - this is true for two colors, but it remains an open problem for more than two.

Updating colorings 2.

- Coloring everything blue is not the best starting point if instead of the exponential λ^x, we choose some arbitrary increasing function $F(x)$.
- If all blue is the best starting point for the exponential (or a broader class of increasing functions), some other property, like convexity, must be behind it.
- Related problems include asking for the probability of a specific vertex being colored blue at the end, etc.

More formal statement of the setting

- Suppose a finite, connected graph $G = (V, E)$. Our random coloring process is a Markov chain (X_t), with state space Q^V, where Q is a finite set of colors. Let F be an increasing function on $\mathbb{N}$.
- At time t, the chain moves as follows: pick a vertex v, and for a coloring $x \in Q^V$ take the set

$$S(x, v) = \left\{ y \in Q^V : \; y(w) = x(w) \; \forall \; w \neq v \right\}.$$

Then, the transition probability $P(x, y) = \boldsymbol{P}[X_{t+1} = y | X_t = x]$ is

$$P(x, y) = \begin{cases} \frac{F\left(N^x_{y(v)}(v)\right)}{Z(x, v, F)} & \text{if } y \in S(x, v) \\ 0 & \text{else,} \end{cases}$$

where $N^x_y(v)$ is the number of the neighbors of v with color $y(v)$ in x, and $Z(x, v, F) = \sum_{q \in Q} F\left(N^x_q(v)\right)$.

Formal statement of the problem

Problem:

For which functions F is it true that, for any choice of the graph G, stopping time T and the sequence $\{v_1, \dots v_T\}$ we have

$$\boldsymbol{P}[X_T = b | X_0 = b] \geq \boldsymbol{P}[X_T = b | X_0 = x],$$

for all $x \in Q^V$, (where b is the all blue coloring)?

Second project: Words solving infinite mazes

- Suppose we have a robot trying to solve a maze M, where a maze is a subgraph of an $n \times n$ grid.
- The robot moves according to a sequence W of instructions (Up, Down, Right, Left), if it can, and stays still otherwise.
- A word W solves M if by following W the robot reaches every vertex of M eventually.

Problem:

Is there an infinite word W that simultaneously solves every maze in the infinite square grid $\mathbb{Z} \times \mathbb{Z}$?

Thank you!