High-School Partitions

REU 2020 project

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COMBINATORIAL STRUCTURES AND PROCESSES
RESEARCH AND INNOVATION STAFF EXCHANGE PROJECT

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Legal partition

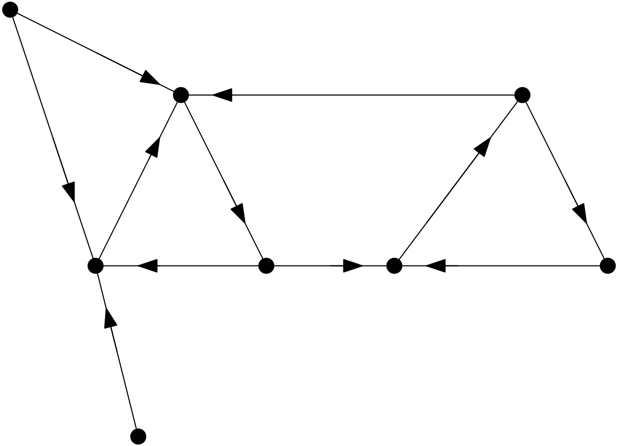
- $G(V, E)$ directed graph
- V is a set of students
- E means preferences:
  - $u \to v \in E$ iff u wants to be with v in the same class

Definition

A partition $V = V_1 \cup \cdots \cup V_k$ into non-empty disjoint sets is called *legal* if

- for each i the subdigraph induced by V_i has all vertices of outdegree at least 1
- in other words, each student has at least one friend in the same class

An example of legal partition



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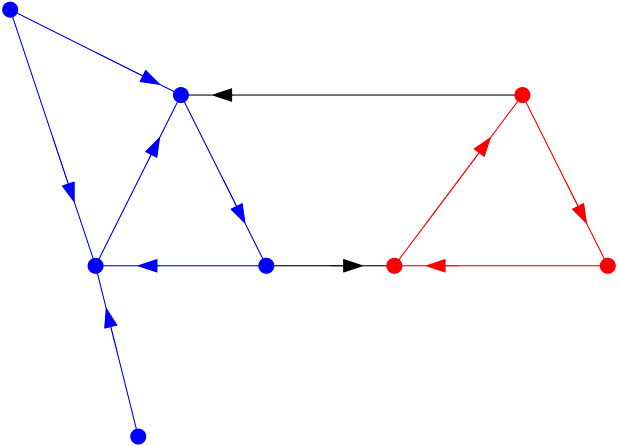

Existence of legal partition into 2 parts

- we focus on partitioning into 2 parts
- the existence of legal partitions for certain digraphs follows from a result of Thomassen:

Theorem

For every digraph with out-degree at least 3 there exists a non-trivial legal partition.

Goal: Explore The Space of Legal Partitions

How many legal partitions are there?

Let $t(d,n)$ denote the minimum number of non-trivial legal partitions of a digraph with n vertices and minimum out-degree at least d. Provide lower a upper bound on $t(d,n)$.

- $t(2,n) = 0$
- $t(3,n) \geq 1$ (Thomassen theorem)
- $t(3,n) \geq 2$ (our result - once we have one legal partition we can find another)
- $\lim_{n\to\infty} t(d,n) = \infty$?

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Let $t(d, n)$ denote the minimum number of non-trivial legal partitions of a digraph with n vertices and minimum out-degree at least d. Provide lower a upper bound on $t(d, n)$.

Can any pair of vertices be separated legally?

Let $G = (V, E)$ be a digraph so that each vertex has out-degree at least d and let $u, v \in V$ be distinct. Does there exist a legal partition $V = V_1 \cup V_2$ such that $u \in V_1, v \in V_2$?

Separability

Definition ($\Phi(d, s)$)

Let $\Phi(d, s)$ denote the following assertion: **For every digraph with out-degree $\geq d$, any subset of s vertices in it can be separated non-trivially by a legal partition.**

- $\Phi(d, s)$ is false iff there exists a digraph with out-degree $\geq d$ and s vertices in it that can't be separated by any legal partition
- $\Phi(d, s) = T \implies \Phi(d + 1, s) = T$
- $\Phi(d, s) = T \implies \Phi(d, s + 1) = T$
- $\Phi(d, s) = T \implies t(d, n) \in \Omega(\log(n))$

Our results - relations of $\Phi(d, s)$

$\Phi(d, d) = F \implies \lim_{n\to\infty} t(d, n) < \infty$

If there is a digraph with out-degree d with d vertices that can't be separated then there are infinitely many digraphs with out-degree d and a constant number of legal partitions.

This implies a dichotomy: for every fixed d and $n \to \infty$ exactly one of the following holds:

- $t(d, n) = O(1)$, or
- $t(d, n) = \Omega(\log(n))$.

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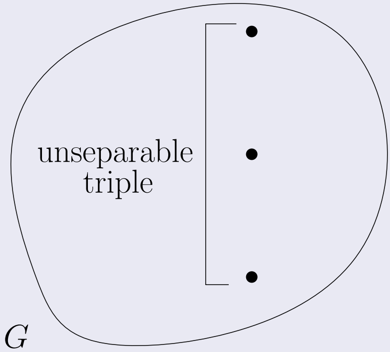
Proof.



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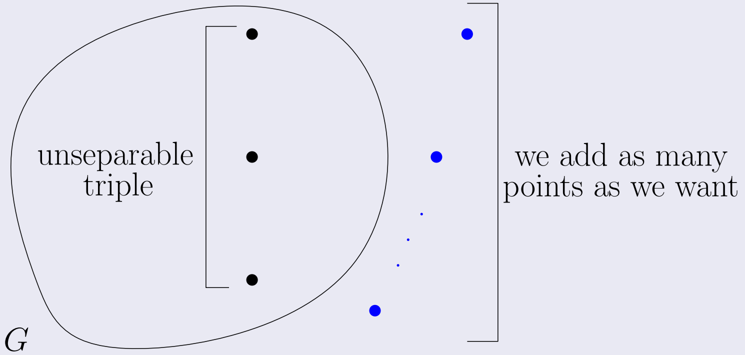

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If there is a digraph with out-degree *d* with *d* vertices that can't be separated then there are infinitely many digraphs with out-degree *d* and a constant number of legal partitions.

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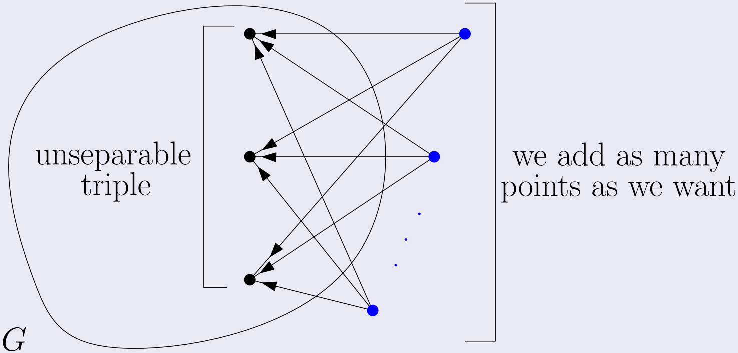

Our results - relations of $\Phi(d, s)$

$\Phi(d,s) = T \implies \Phi(d, d-1) = T$

If we can separate each s-tuple in each digraph of out-degree at least d then we can separate each $(d-1)$-tuple as well.

$\Phi(3,s) = T \implies \Phi(3,2) = T$

Consequently, if we are able to separate each s-tuple in each digraph of out-degree at least 3 for s arbitrary, we can also separate each pair of vertices.

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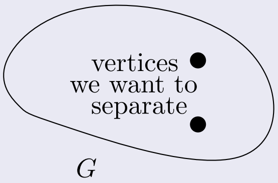

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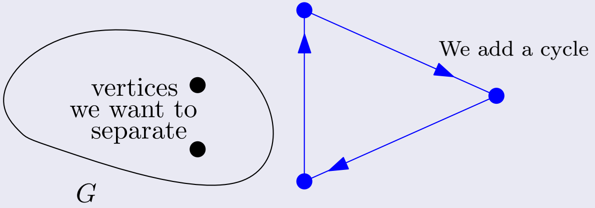

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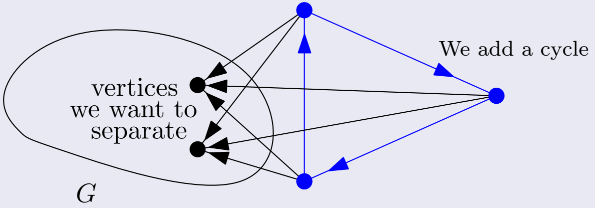

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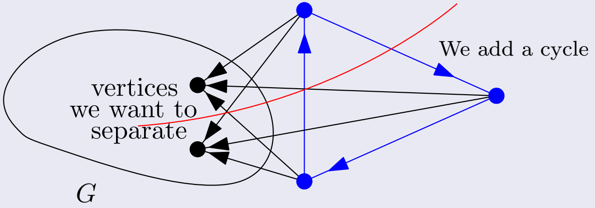


Our results - relations of $\Phi(d, s)$

$\Phi(d, s) = T \implies \Phi(d + 1, 2)$

If we can separate each s-tuple of vertices in each digraph of out degree at least d then we can separate each pair of vertices in each digraph of out-degree at least $d + 1$

Our results - regular digraphs

Definition

A digraph is d-regular if all vertices have out-degree and in-degree d.

Using Lovász local lemma we get following.

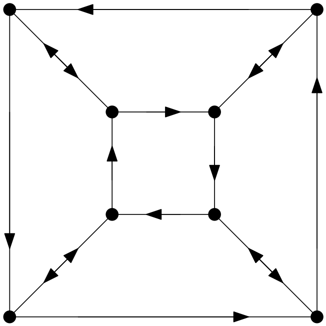
Theorem

In each 9-regular digraph all pairs of vertices are separable.

Our results - Transitive digraphs

Definition

A digraph $G = (V, E)$ is vertex-transitive if for every $u, v \in V$ there exists an automorphism φ such that $\varphi(u) = v$.



Our results - Transitive digraphs

Theorem

Let $G = (V, E)$ be a transitive digraph such that $|V|$ is prime. If there exists a legal partition then every pair of vertices is separable.

Corollary

Let $G = (V, E)$ be a transitive digraph such that $|V|$ is prime and the outdegree is at least 3. All pairs of vertices in such digraph are separable.

Theorem

Let $G = (V, E)$ be a transitive digraph with outdegree 3. If $u, v \in V$ are unseparable vertices then there is no edge between them.