Antipodal monochromatic paths in hypercubes

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Hypercubes

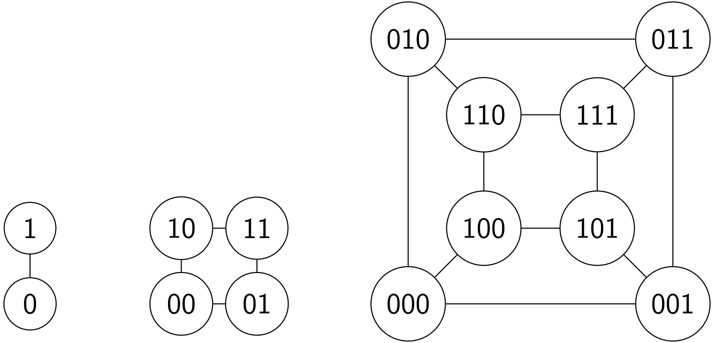

Figure 1: From left to right, graphs Q_1, Q_2 and Q_3.

Definition

The n-dimensional hypercube Q_n is an undirected graph with $V(Q_n) = \{0, 1\}^n$ and $E(Q_n) = \{(u, v) : u \text{ and } v \text{ differ in exactly one coordinate}\}$.

Antipodal vertices

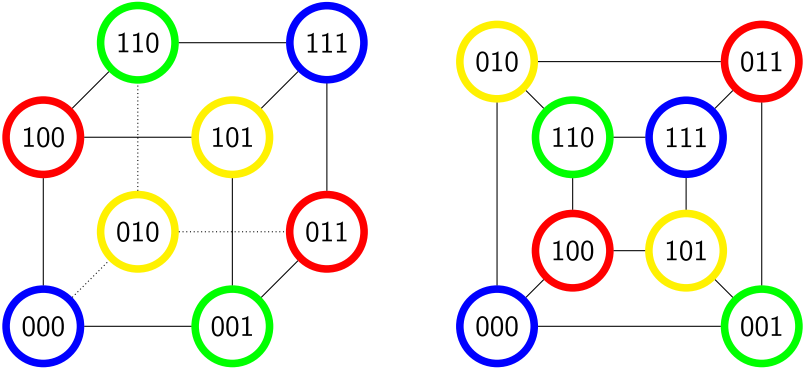

Figure 2: Antipodal vertices of a Q_3 are drawn with the same color.

Definition

Let u be a vertex of the hypercube Q_n. Its *antipodal vertex* u' is the vertex which differs from u in every coordinate.

Antipodal edges

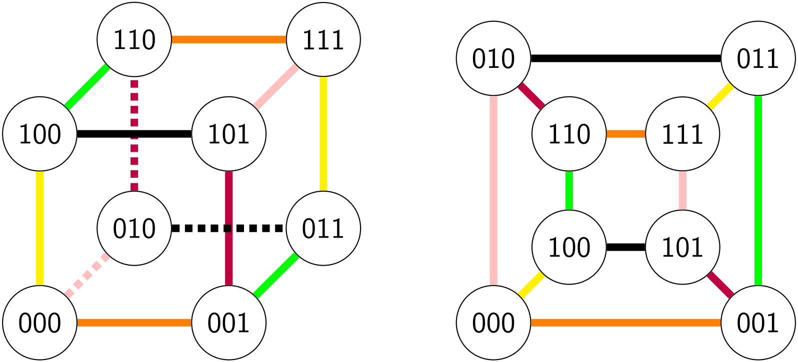

Figure 3: Antipodal edges of a Q_3 are drawn with the same color.

Definition

Let $e = (u, v)$ be an edge of the hypercube Q_n. Its *antipodal edge* is the edge $e' = (u', v')$.

Colorings

Definition

An *edge* 2-*coloring* is any mapping $c : E(Q_n) \to \{\text{red}, \text{blue}\}$.

From now on, by a 2-coloring we always mean edge 2-colorings.

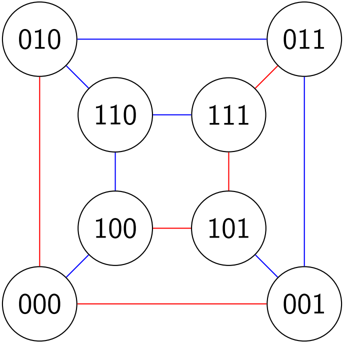

Figure 4: A 2-coloring of Q_3.

All kinds of paths

Definition

A path is *monochromatic*, if all its edges have the same color.

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- Let $e = \{u, v\} \in E(Q_n)$ be an edge of a hypercube. If $u \oplus v$ has its sole 1 in the i-th coordinate, the we say that e is in i-th *direction*.
- In hypercubes, directions of edges of any geodesic are pairwise different.

A natural question

Question

Given *any* 2-coloring of a Q_n, is there always a pair of antipodal vertices such that there is a monochromatic path connecting them?

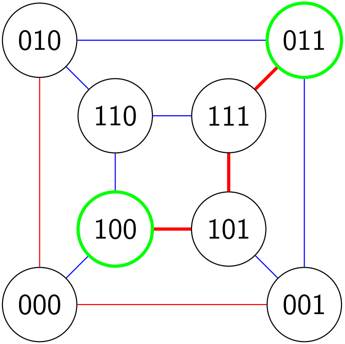

Figure 5: A 2-coloring of Q_3, a monochromatic path between green antipodal vertices is drawn by a thicker line.

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Answer

No, see Figure 6.

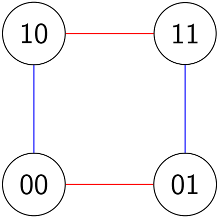

Figure 6: A counterexample where there's no monochromatic path between any antipodal pair.

Antipodal colorings

Definition

A 2-coloring is *antipodal* if all pairs of antipodal edges have different colors.

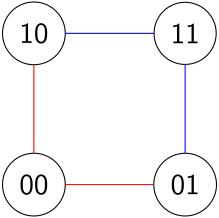

Figure 7: The only antipodal coloring of Q_2 (up to isomorphism).

Conjecture (S. Norine [Nor08])

For any antipodal coloring of a hypercube Q_n there always exists a pair of antipodal vertices such that there is a monochromatic path connecting them.

Switches on a path

Definition

A *switch* on a path $P = (u_1, \ldots, u_\ell)$ occurs at vertex u_i if edges of path P incident to u_i have different colors. A *k-switch* path is a concatenation of $k+1$ monochromatic paths.

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The *number of switches* between vertices $u, v \in G$ is the least number k such that there is a k-switch path between them.

Norine's conjecture restated: Is there always a 0-switch path between some pair of antipodal vertices of Q_n for all antipodal colorings?

One switch conjecture

Conjecture (Feder and Subi [FS13])

For any coloring[1] of a hypercube Q_n there always exists a pair of antipodal vertices such that there is a 1-switch path connecting them.

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- It is known that if this conjecture holds, then it implies Norine's conjecture.
- "One switch conjecture" and its lack of antipodal colorings are more amenable to inductive proofs, as we do not have a global restriction on the coloring.

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History

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- These results show that there is always a pair of antipodal vertices such that there is a $\frac{n}{2}$-switch path, resp. geodesic, connecting them.
- Dvořák improves the bound of Leader and Long to show that there is always a pair of antipodal vertices such that there is a $\left(\frac{3}{8} + o(1)\right)n$-switch geodesic connecting them.

Dvořák's approach

Theorem (Dvořák [Dvo19])

Theorem 5. *In any 2-coloring of edges of Q_n, we can find a pair of antipodal vertices and a geodesic joining them with at most $\left(\frac{3}{8} + o(1)\right)n$ switches.*

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Main point of interest

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- Dvořák's approach requires information about the average number of switches inside hypercubes over 2-colorings.
- Which naturally leads to the following definition...

Definition

Let $\mathcal{C}_n$ be the set of all colorings of Q_n. Let $\#_c(u, v)$ be the number of switches of the least switch geodesic between vertices u and v with respect to coloring $c \in \mathcal{C}_n$. We define function $f\colon \mathbb{N} \to \mathbb{R}_0^+$ as the maximum average number of switches between antipodal pairs of vertices of Q_n over all 2-colorings of Q_n, that is

$$f(n) = \max_{c \in \mathcal{C}_n} \frac{\sum_{u \in V(Q_n)} \#_c(u, u')}{2^n}.$$

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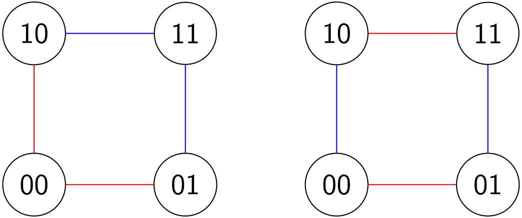

Figure 8: Two colorings of Q_2 with a nonzero switch vector. The left cube has switches $0, 1$ and the right cube has switches $1, 1$.

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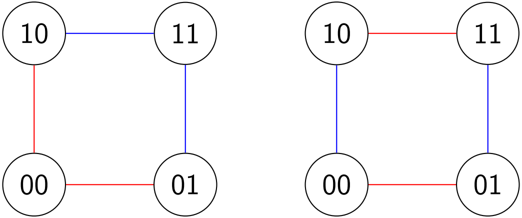
- $f(3) = 1$ by Dvořák [Dvo19].



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- function f is nondecreasing,
- $f(4) = \frac{5}{4}$ and we know all colorings with 10 switches,
- $f(5) \geq f(4) = \frac{5}{4}$,
- in some special cases $f(5) = \frac{5}{4}$.

A taste of our techniques

Theorem

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Proof

- Take any coloring c of Q_n which corresponds to the maximum value $f(n)$.

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- Take any coloring c of Q_n which corresponds to the maximum value $f(n)$.
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- Color the remaining crossing edges arbitrarily.

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- Consider any antipodal geodesic $P = (x = v_0, \ldots, v_{n+1} = x')$ and let $\{v_j, v_{j+1}\}$ be its sole crossing edge.

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- Observe that if we perform steps in the same directions as P one after another, but we skip the sole crossing edge, then the number of switches on such path can only decrease.
- Perform this analysis for each antipodal geodesic to obtain the result. $\square$

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- It holds that $f(5) = \frac{5}{4}$.
  - We did not find a counterexample :)
- For $n \in \mathbb{N}$ it holds that $f(2n) = f(2n+1)$.
  - Recall that $f(2) = f(3) = 1$ and $f(4) = \frac{5}{4}$.

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