Shannon capacity and the categorical product

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Categorical product of graphs

For two graphs F and G, we define $F \times G$ as follows.

$$V(F \times G) = V(F) \times V(G)$$

and

$$E(F \times G) = \{(a,b)(c,d) : ac \in E(F), bd \in E(G)\}.$$

Observe: If $c : V(F) \to [k]$ is a proper coloring of F, then $f_c : (a,b) \mapsto c(a)$ is a proper coloring of $F \times G$, implying $\chi(F \times G) \leq \chi(F)$. Similarly, $\chi(F \times G) \leq \chi(G)$.

Hedetniemi's conjecture

Thus we have $\chi(F \times G) \leq \min\{\chi(F), \chi(G)\}$, and Hedetniemi conjectured in 1966, that in fact, for every two finite simple graphs F and G

$$\chi(F \times G) = \min\{\chi(F), \chi(G)\}.$$

It is trivial for $\mathrm{RHS} = \mathbf{2}$, is an easy exercise for $\mathrm{RHS} = \mathbf{3}$, a difficult theorem (El-Zahar - Sauer 1985) for $\mathrm{RHS} = \mathbf{4}$, open for $\mathrm{RHS} = \mathbf{5}$, and refuted in general very recently by Yaroslav Shitov, `arxiv.org`, May 6, 2019.

Recent progress on Hedetniemi's conjecture

Y. Shitov: Counterexamples to Hedetniemi's conjecture, (arxiv, May 6, 2019) presents a family of counterexamples.

C. Tardif, X. Zhu: A note on Hedetniemi's conjecture, Stahl's conjecture and the Poljak-Rödl function (arxiv, June 10, 2019) shows that the additiv gap is arbitrarily large.

X. He, Y. Wigderson: Hedetniemi's conjecture is asymptotically false (arxiv, June 23, 2019) shows that the ratio of the two sides in the conjecture is asymptotically also away from $\mathbf{1}$.

Topological connections

H Hajiabolhassan, F Meunier: Hedetniemi's conjecture for Kneser hypergraphs, JCTA 2016, proves the conjecture for topologically relevant special cases.

T. Matsushita 2019, M. Wrochna 2019 independently proves that a consequence of Hedetniemi's conjecture is/would have been the following topological statement.

If $n \in N$ and X, Y are Z_2-spaces (finite Z_2-simplicial complexes) such that $X \times Y$ admits a Z_2-map to the n-dimensional sphere, then X or Y itself admits such a map.

More general framework

Let $p(G)$ be any graph parameter satisfying that $p(G) \leq p(H)$ whenever $G \to H$, i.e., when a graph homomorphism exists from G to H.
($f : V(G) \to V(H)$ is a graph homomorphism if it preserves edges.)

We have $F \times G \to F$ and $F \times G \to G$ (by the projection maps), so for a $p(G)$ as above one always has

$$p(F \times G) \leq \min\{p(F), p(G)\}.$$

It is thus natural to ask for any such parameter $p(G)$ whether equality holds.

Sometimes equality holds trivially, e.g., for the clique number:

We obviously have

$$\omega(F \times G) = \min\{\omega(F), \omega(G)\}.$$

Two nontrivial examples when equality holds:

Theorem (Zhu 2011) For the fractional chromatic number χ_f and any two finite simple graphs F and G one has

$$\chi_f(F \times G) = \min\{\chi_f(F), \chi_f(G)\}.$$

Theorem (Godsil, Roberson, Šamal, Severini 2016): For the Lovász theta number of the complementary graph, $\bar{\vartheta}$, and any two finite simple graphs F and G one has

$$\bar{\vartheta}(F \times G) = \min\{\bar{\vartheta}(F), \bar{\vartheta}(G)\}.$$

Shannon (OR-)capacity

Def. The OR-product $F \cdot G$ of two graphs F and G is defined on vertex set $V(F \cdot G) = V(F) \times V(G)$ with edge set

$$\{(f,g)(f',g') : ff' \in E(F) \text{ or } gg' \in E(G)\}.$$

Denoting by G^t the t-fold OR-product of G with itself, Shannon OR-capacity is defined as

$$C_{\mathrm{OR}}(G) := \lim_{t \to \infty} \sqrt[t]{\omega(G^t)}.$$

Shannon OR-capacity is also homomorphism monotone:

$$F \to G \Rightarrow C_{\mathrm{OR}}(F) \leq C_{\mathrm{OR}}(G).$$

Thus we have

$$C_{\mathrm{OR}}(F \times G) \leq \min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\},$$

and it makes sense to ask whether equality holds or to ask for what graphs it holds. (Note that $C_{\mathrm{OR}}(G) \leq \bar{\vartheta}(G) \leq \chi_f(G)$, i.e., both of our examples are well-known upper bounds on Shannon OR-capacity.)

Some counterevidence for equality

$C_{\mathrm{OR}}(G)$ does not behave "well" for other operations.

1. Haemers (1979) showed that $\exists$ graphs G, H s.t.

$$C_{\mathrm{OR}}(G \cdot H) \neq C_{\mathrm{OR}}(G) C_{\mathrm{OR}}(H).$$

This answered a question of Lovász (1979) in the negative, who noted that $C_{\mathrm{OR}}(G \cdot H) \geq C_{\mathrm{OR}}(G) C_{\mathrm{OR}}(H)$ obviously holds.

2. Let $G \oplus H$ denote the *join* of graphs G and H, i.e., a vertex-disjoint copy of G and H with all the edges between the two parts.

It was shown by Alon (1998), that $\exists$ graphs G and H s.t.

$$C_{\mathrm{OR}}(G \oplus H) \neq C_{\mathrm{OR}}(G) + C_{\mathrm{OR}}(H),$$

although equality was conjectured by Shannon (1956), who proved that $C_{\mathrm{OR}}(G \oplus H) \geq C_{\mathrm{OR}}(G) + C_{\mathrm{OR}}(H)$ always holds. The question of equality was open for more than 40 years.

One of our main contributions is the observation that the counterevidence presented above is rather weak. It should be emphasized though that it may be too much to interpret this as "supporting evidence".

General upper bounds on $C_{\mathrm{OR}}(G)$

Let G be a graph parameter $p(G)$ satisfying

$\omega(G) \le p(G)$ and

$p(F \cdot G) \le p(F)p(G)$ for any F and G.

Then $C_{\mathrm{OR}}(G) \le p(G)$, because

$$C_{\mathrm{OR}}(G) = \lim_{t\to\infty} \sqrt[t]{\omega(G^t)} \le \lim_{t\to\infty} \sqrt[t]{p(G^t)} \le$$

$$\lim_{t\to\infty} \sqrt[t]{[p(G)]^t} = p(G).$$

χ_f and $\bar{\vartheta}$ are both of this type, but they have even stronger properties:

$$\chi_f(F \cdot G) = \chi_f(F)\chi_f(G),$$
$$\bar{\vartheta}(F \cdot G) = \bar{\vartheta}(F)\bar{\vartheta}(G).$$

They also satisfy

$$\chi_f(F \oplus G) = \chi_f(F) + \chi_f(G),$$
$$\bar{\vartheta}(F \oplus G) = \bar{\vartheta}(F) + \bar{\vartheta}(G).$$

Let $\boldsymbol{X}$ be the class of all mappings φ from graphs to the nonnegative real numbers that satisfy

1. $\boldsymbol{\varphi(K_1) = 1}$
2. $\boldsymbol{\varphi(F \oplus G) = \varphi(F) + \varphi(G)}$
3. $\boldsymbol{\varphi(F \cdot G) = \varphi(F)\varphi(G)}$
4. $\boldsymbol{F \to G \Rightarrow \varphi(F) \leq \varphi(G)}$.

$\boldsymbol{X}$ is called the asymptotic spectrum of graphs. Using Strassen's theory of asymptotic spectra, Zuiddam (2018+) obtained a surprising characterization of Shannon capacity.

Zuiddam's recent result

Note that $\varphi(K_1) = 1$ and $\varphi(F \oplus G) = \varphi(F) + \varphi(G)$ implies $\varphi(K_n) = n$. Thus for any $\varphi \in X : C_{\mathrm{OR}}(G) \leq \varphi(G)$.

At the same time $C_{\mathrm{OR}}(G) \notin X$ by the mentioned results of Haemers (1979) and Alon (1998).

Theorem (Zuiddam 2018):

$$C_{\mathrm{OR}}(G) = \min_{\varphi \in X} \varphi(G).$$

Known elements of X.

Zuiddam lists four types of elements of X:
Two of them are χ_f and $\bar{\vartheta}$, one is called *fractional orthogonal rank* and the fourth one is the so-called *fractional Haemers bound* (introduced by Anna Blasiak (2013)) that also depends on a chosen field thus providing infinitely many examples. (A result of Bukh and Cox (2019) shows that they are different, indeed, in an essential sense.)
It would be interesting to know whether these other parameters also satisfy a Hedetniemi type equality as χ_f and $\bar{\vartheta}$ do.

Theorem:

Either $\exists \varphi \in \boldsymbol{X}$ for which there are graphs $\boldsymbol{F}$ and $\boldsymbol{G}$ s.t.

$$\boldsymbol{\varphi(F \times G) \neq \min\{\varphi(F), \varphi(G)\},}$$

or we also have

$$\boldsymbol{C_{\mathrm{OR}}(F \times G) = \min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\}}$$

for every $\boldsymbol{F}$ and $\boldsymbol{G}$.

The proof is a straightforward consequence of Zuiddam's theorem:

$$\boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{\varphi}(\boldsymbol{F} \times \boldsymbol{G})$$

for some $\boldsymbol{\varphi} \in \boldsymbol{X}$. However, by $\boldsymbol{\varphi} \in \boldsymbol{X}$ we have

$$\mathbf{min}\{\boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{F}), \boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{G})\} \leq \mathbf{min}\{\boldsymbol{\varphi}(\boldsymbol{F}), \boldsymbol{\varphi}(\boldsymbol{G})\},$$

so if $\boldsymbol{\varphi}(\boldsymbol{F} \times \boldsymbol{G}) = \mathbf{min}\{\boldsymbol{\varphi}(\boldsymbol{F}), \boldsymbol{\varphi}(\boldsymbol{G})\}$, then it also implies $\boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{F} \times \boldsymbol{G}) \geq \mathbf{min}\{\boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{F}), \boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{G})\}$, while the opposite inequality also holds. $\square$

Test cases?

For two graphs T and Z, let $T_Z \subseteq T$ be a subgraph of T admitting a homomorphism to Z and having maximum value of Shannon OR-capacity among all those subgraphs.
Proposition:

$$C_{\mathrm{OR}}(F \times G) \geq \max\{C_{\mathrm{OR}}(F_G), C_{\mathrm{OR}}(G_F)\}.$$

Finding examples of graphs where the above lower bound is strictly smaller than the upper bound $\min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\}$ is already a challenge.

Ramsey graphs might provide candidates

Erdős-McEliece-Taylor (1971) proved (rediscovered by Alon and Orlitsky (1995)) that

$$\max\{\omega(G^t) : \omega(G) < k\} = R(k, \ldots, k) - 1,$$

where k appears t times in the brackets of RHS and $R(., \ldots, .)$ is the usual Ramsey number.
This implies

$$\max\{C_{\mathrm{OR}}(G) : \omega(G) < k\} = \limsup_{t\to\infty} \sqrt[t]{R(k, \ldots, k)}.$$

In particular,

$$\max_{K_3 \not\subseteq G}\{C_{\mathrm{OR}}(G)\} = \limsup_{t\to\infty} \sqrt[t]{R(3,\ldots,3)}.$$

(Whether the RHS here is finite is a famous Erdős problem.)

If G is triangle-free, then $K_3 \not\to G$. The Ramsey graph for $R(3,3) = 6$ is C_5, and $C_5 \to K_3$. Thus this does not provide a test case, as by $C_{2\max\{r,\ell\}+1} \to C_{2\min\{r,\ell\}+1}$:

$$C_{\mathrm{OR}}(C_{2r+1} \times C_{2\ell+1}) = \min\{C_{\mathrm{OR}}(C_{2r+1}), C_{\mathrm{OR}}(C_{2\ell+1})\}.$$

However, the Ramsey graph for $\boldsymbol{R(3,3,3) = 17}$, the so-called Clebsch graph $\boldsymbol{B_{16}}$ on $\mathbf{16}$ points is not $\mathbf{3}$-chromatic, i.e., $\boldsymbol{B_{16} \nrightarrow K_3}$. Therefore $\boldsymbol{(B_{16})_{K_3} \subsetneq B_{16}}$.

A further difficulty is that $\boldsymbol{C_{\mathrm{OR}}(B_{16})}$ is not known. We only know

$$\boldsymbol{\sqrt[3]{16} \le C_{\mathrm{OR}}(B_{16}) \le \bar{\theta}(B_{16}) = 8/3.}$$

Yet, it might shed some light on our question if we knew whether

$$\boldsymbol{C_{\mathrm{OR}}(B_{16} \times K_3) \ge \sqrt[3]{16}.}$$

To prove at least that there are graph pairs where our easy lower and upper bounds on $C_{\mathrm{OR}}(F \times G)$ do not coincide an affirmative answer to the following question would be enough:

Question: Is it true that if G is K_3 -free and $\chi(G) = 3$, then $\bar{\theta}(G) < 3$?

Since $\limsup_{t\to\infty} R(3,\ldots,3) > 3.1717 > 3$ is known by a result of Exoo (1994), a K_3-free graph G with $C_{\mathrm{OR}}(G) > 3$ exists. An affirmative answer to the above question would mean that for this G we have $C_{\mathrm{OR}}(G_{K_3}) < 3$, and thus our upper and lower bounds would differ for $C_{\mathrm{OR}}(G \times K_3)$.

Remark: $\exists G$ s.t. it is K_3 -free and $\chi_f(G) = \chi(G) = 3$, so we cannot change $\bar{\theta}$ to χ_f in the previous argument.

Example: $V(G) := \{0,1\}^3 \cup \{a,b,c,d\}$ and let $G[\{0,1\}^3] \cong 1-\text{skeleton of the cube}$ while a,b,c,d each connect to a different antipodal pair of $\{0,1\}^3$ and we also have the edges ab and cd. Then

$$|V(G)| = 12, \alpha(G) = 4, \chi(G) = \chi_f(G) = 3.$$

Another candidate

Let $F = K_4$ and $G = H_{17}$ be the well-known Ramsey graph on 17 vertices containing no clique or independent set of size 4.

Known: $C_{\mathrm{OR}}(H_{17}) = \sqrt{17}$ (by results of Lovász) and $C_{\mathrm{OR}}(K_4) = 4$, thus $\min\{C_{\mathrm{OR}}(K_4), C_{\mathrm{OR}}(H_{17})\} = 4$.
We also have $F_G = K_3$ and $G \not\to F$ in this case. In fact, $|V(G_F)| \leq 12$, and if $C_{\mathrm{OR}}(G_F) < 4$ were true, then our lower bound would be strictly smaller than the upper bound.

So answering the question

$$\boldsymbol{C}_{\mathrm{OR}}(\boldsymbol{H}_{\mathbf{17}} \times \boldsymbol{K}_4) = \mathbf{4?}$$

might shed some light on our problem.

Open questions summarized

$\mathrm{Q1}$ **:** $\quad \boldsymbol{\exists G, H : C_{\mathrm{OR}}(G \times H) < \min\{C_{\mathrm{OR}}(G), C_{\mathrm{OR}}(H)\}?}$

Q2:

$\exists \varphi \in \boldsymbol{X}$ and graphs $\boldsymbol{G, H}\ \ \boldsymbol{\varphi(G{\times}H) < \min\{\varphi(G), \varphi(H)\}?}$

We have seen that we cannot have a positive answer to Q1 without a positive answer to Q2.

What is the answer to Q2 if φ is the fractional Haemers bound?

Q3: Give two graphs G, H satisfying

$$\max\{C_{\mathrm{OR}}(F_G), C_{\mathrm{OR}}(G_F)\} < \min\{C_{\mathrm{OR}}(G), C_{\mathrm{OR}}(H)\}.$$

Q4: In particular, if F is a 4-chromatic subgraph of H_{17} with largest $C_{\mathrm{OR}}(F)$ value, is then $C_{\mathrm{OR}}(F) < 4$?

Q5:

$$\max\{\bar{\theta}(G) : \omega(G) = 2, \chi(G) = 3\} = ?$$

In particular, is it strictly less than 3?

Q6: Is

$$C_{\mathrm{OR}}(H_{17} \times K_4) = 4?$$

Q7: Is

$$C_{\mathrm{OR}}(B_{16} \times K_3) \geq \sqrt[3]{16}?$$

Q8: Is at least

$$C_{\mathrm{OR}}(G \times K_n) = \min\{C_{\mathrm{OR}}(G), n\}$$

true for all graphs G and positive integers n? Is it true for $n = 3$?