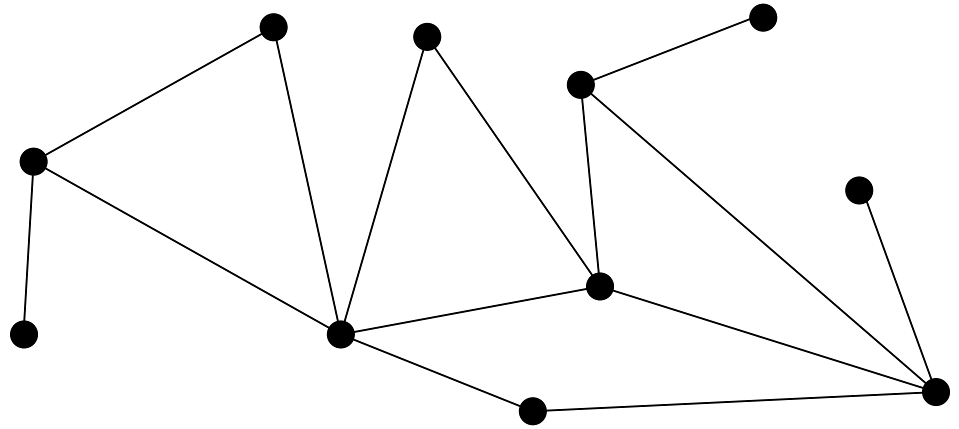
Topological Connectedness and Independent Sets in Graphs

Penny Haxell

University of Waterloo

Independent sets

A set T of vertices in a graph G is independent if no edge of G joins two vertices of T.



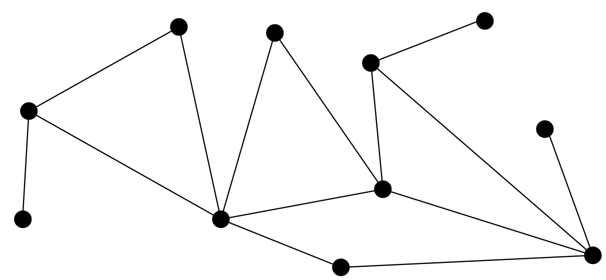
Aim: To see how topological notions can (sometimes!) help us understand the independent sets in a graph.

Simplicial complexes

An abstract simplicial complex is a family Σ of sets (called simplices) such that if B is a subset of $A \in \Sigma$ then B is in Σ. i.e. a "downward-closed hypergraph".

The dimension of Σ is d where $d+1$ is the largest size of a simplex in Σ.

The independence complex $\mathcal{I}(G)$ of a graph G is the abstract simplicial complex consisting of all independent sets of vertices in G.



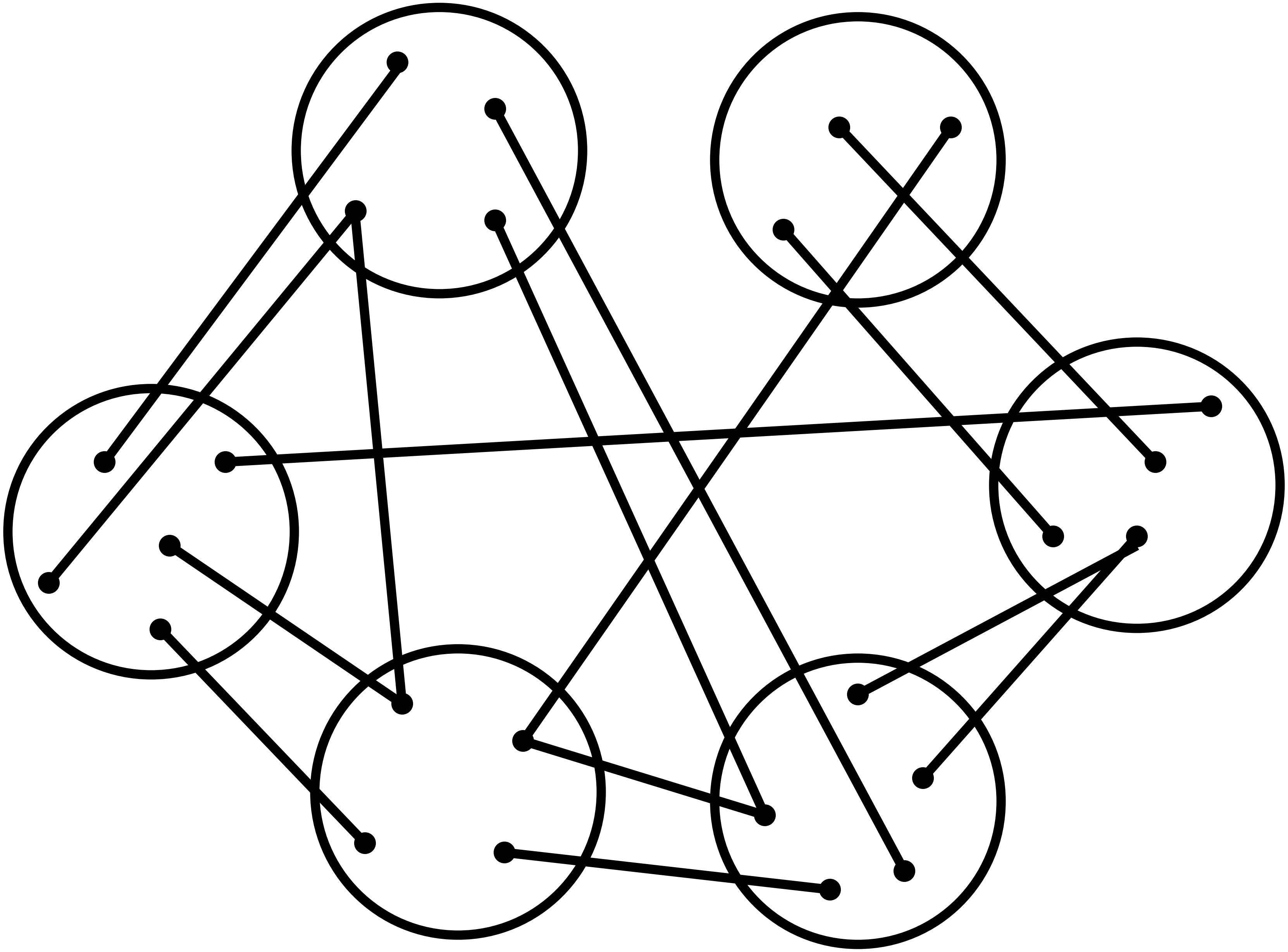
NASTY simplicial complex!

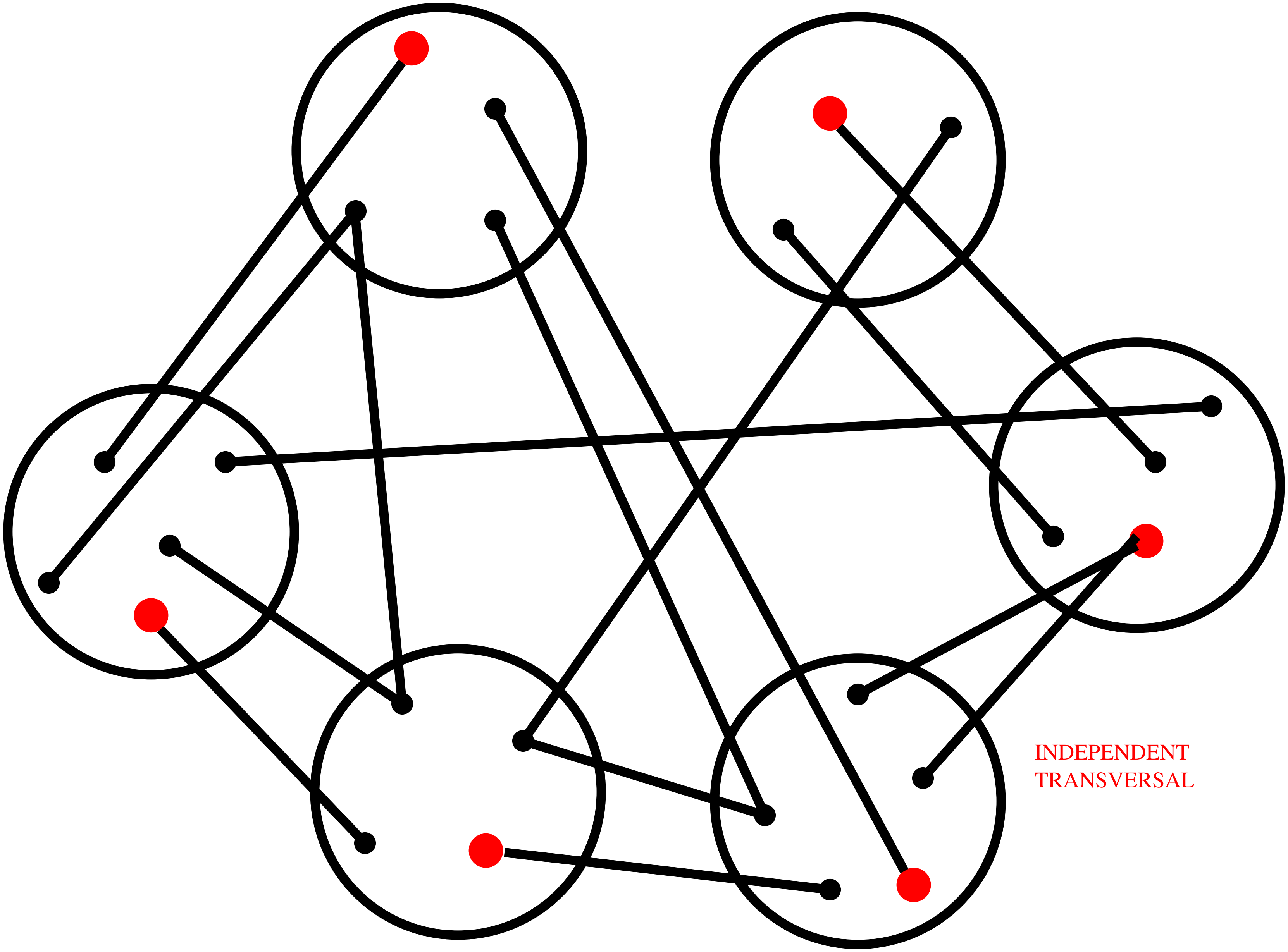
Independent transversals

Let G be a graph with a fixed partition of its vertex set.

An independent transversal in G is a subset T of vertices such that

- T is independent,
- T contains exactly one vertex from each partition class (transversal)

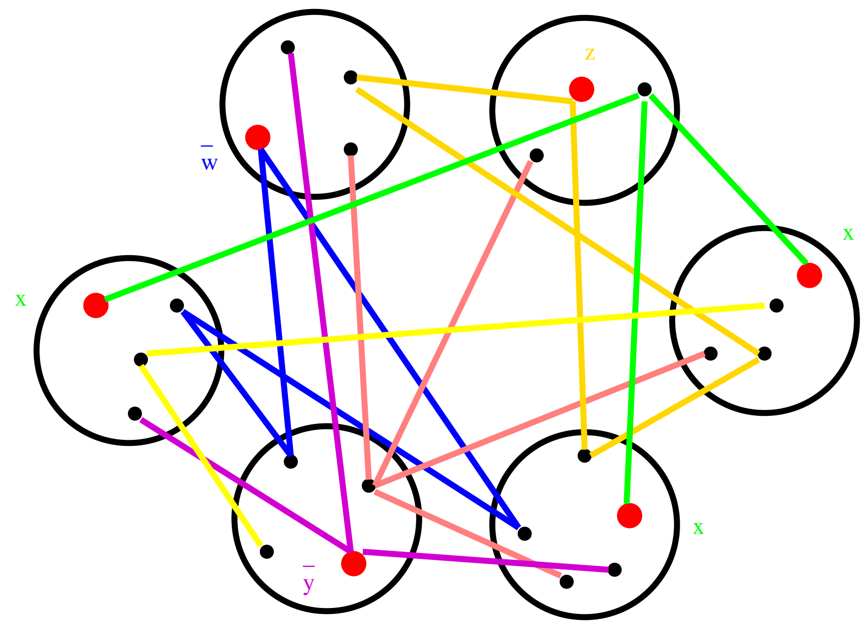



INDEPENDENT
TRANSVERSAL

Independent Transversals

Many combinatorial problems can be formulated by asking whether a given graph with a given vertex partition has an independent transversal. For example, the SAT problem:

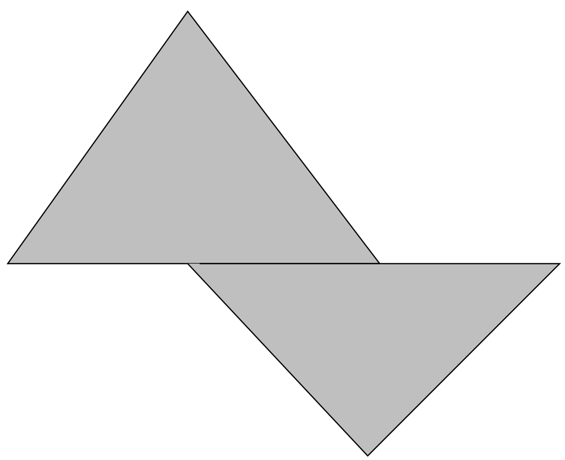
$(x_1 \vee \bar{x_4} \vee x_7) \wedge (\bar{x_1} \vee \bar{x_3} \vee x_2) \wedge (x_3 \vee \bar{x_2}) \wedge (x_5 \vee x_6 \vee \bar{x_2})$



Simplicial complexes

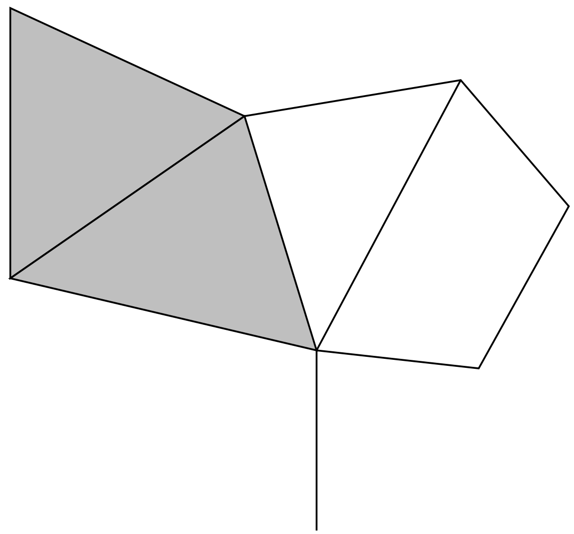
A geometric simplicial complex is a family Δ of simplices in real space such that

- if τ is a face of $\sigma \in \Delta$ then τ is in Δ, and
- the intersection of any two simplices in Δ is a face of both.



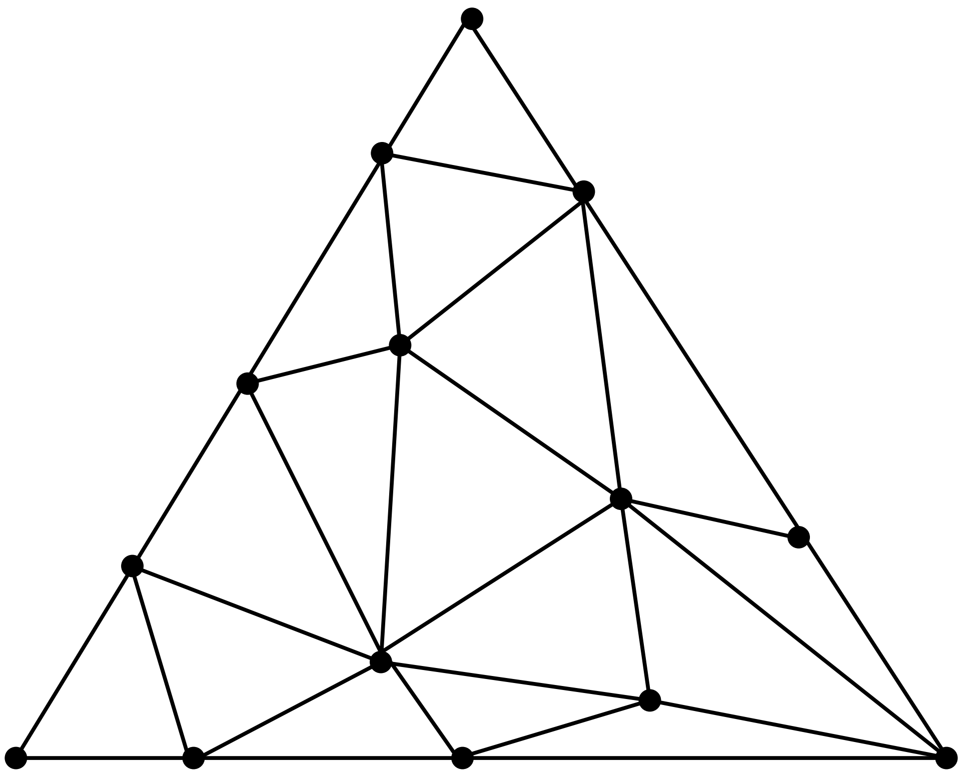
Simplicial complexes

Thus the family of vertex sets of the simplices in a geometric simplicial complex is an abstract simplicial complex.



Conversely, every abstract simplicial complex has a geometric realization.

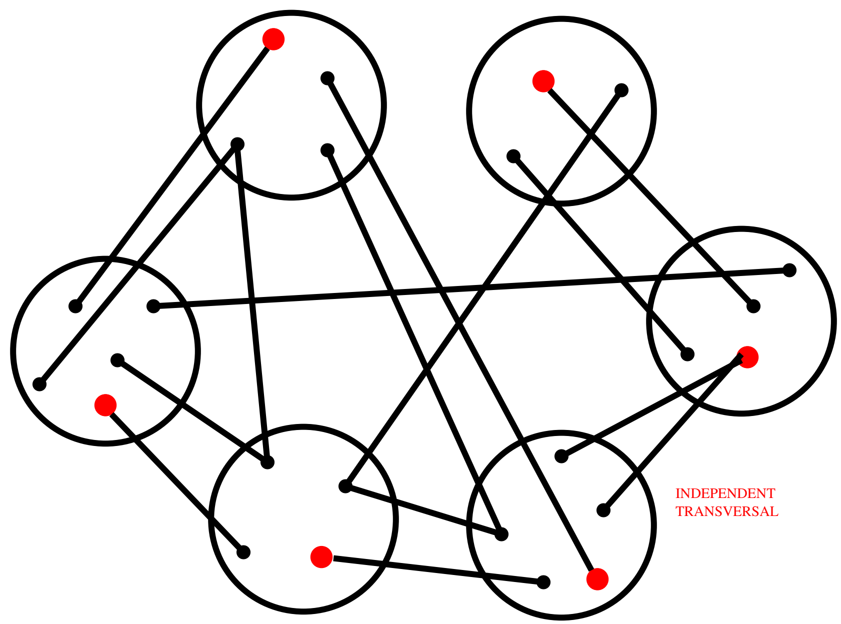
Geometric simplicial complexes



A triangulated n-simplex is a geometric simplicial complex.

NICE simplicial complex!

When does an independent transversal exist?



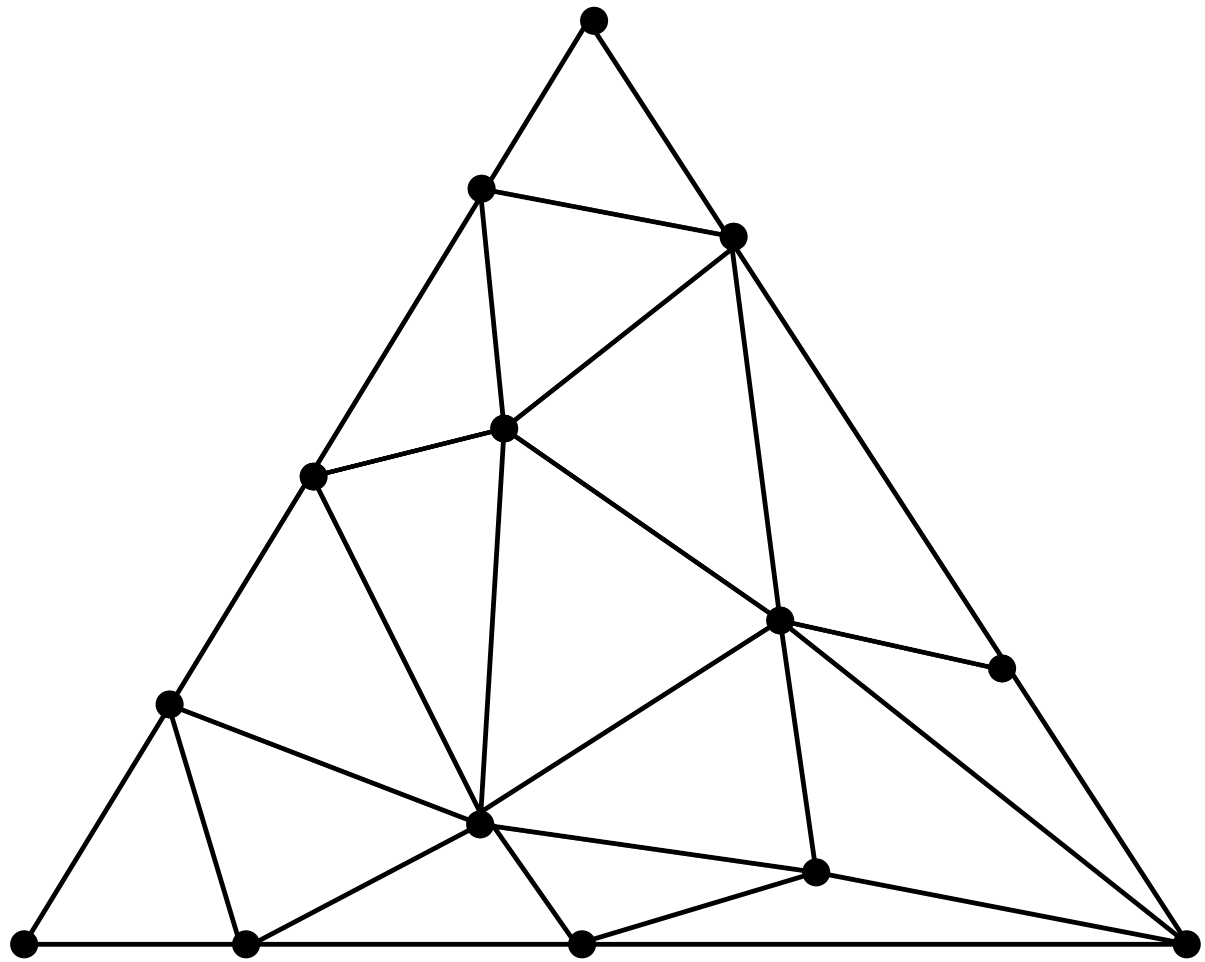
If we view the partition classes as colours for the vertices, then an independent transversal is the same as a multicoloured simplex in the simplicial complex $\mathcal{I}(G)$.

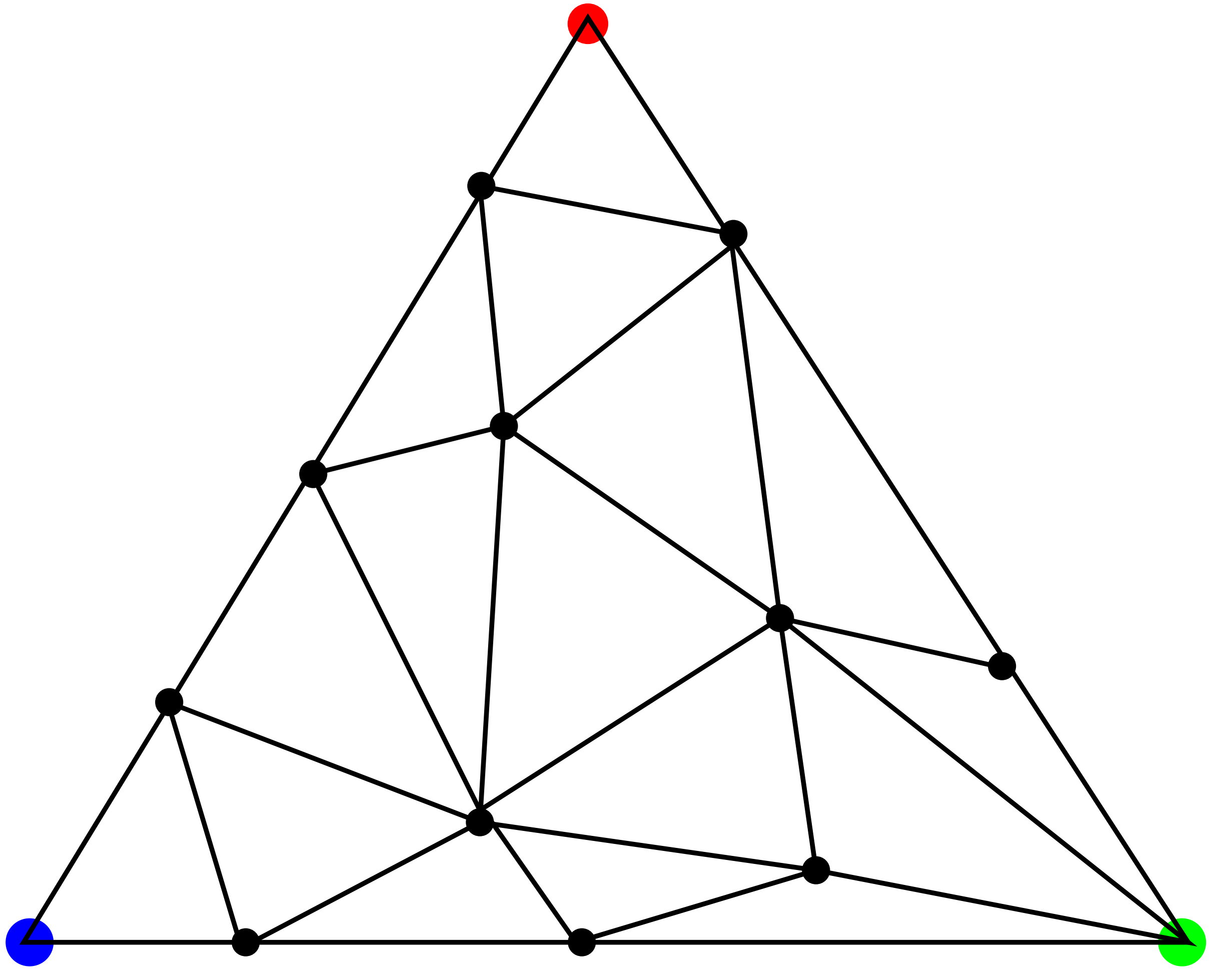
NASTY simplicial complex!

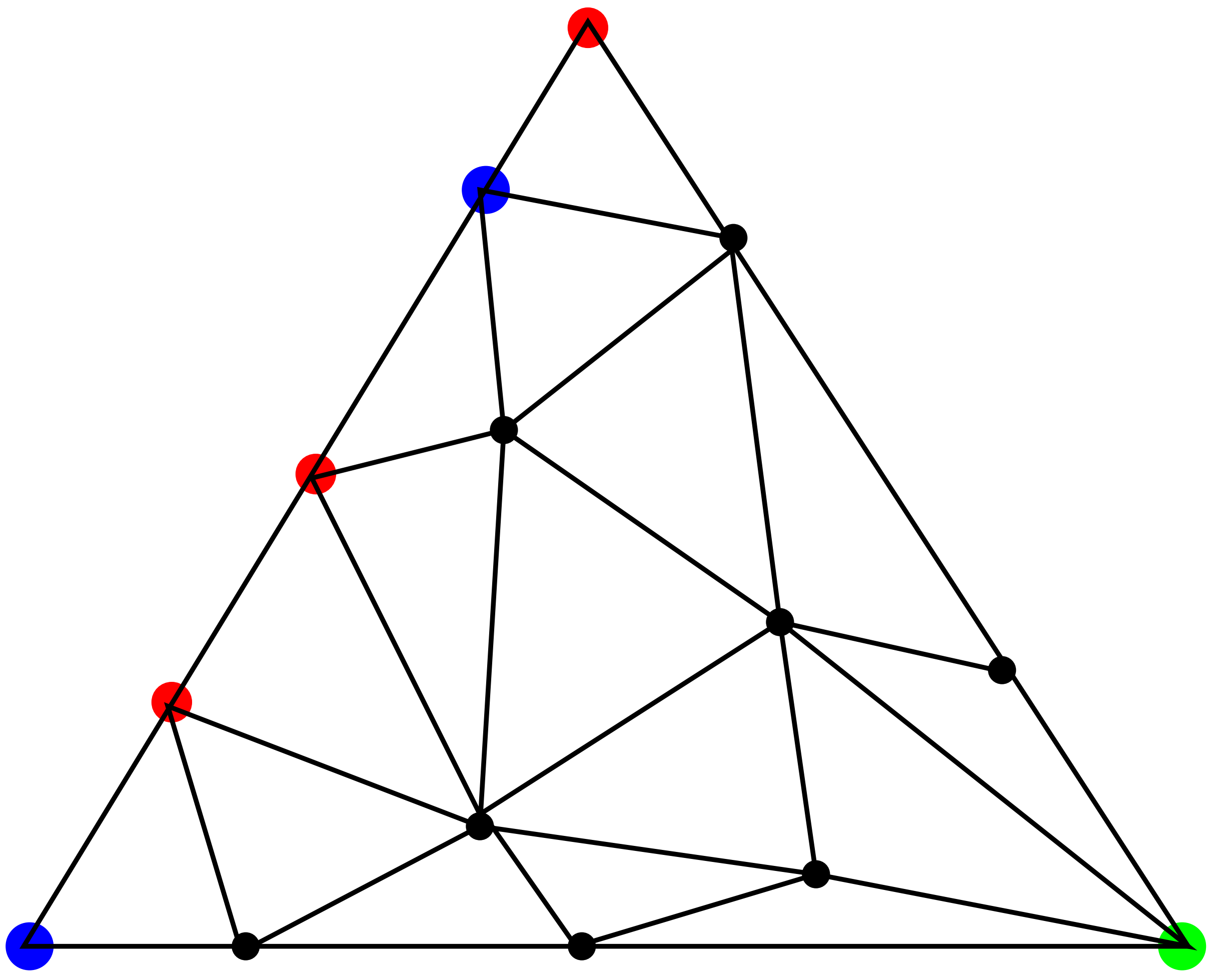
Sperner's Lemma

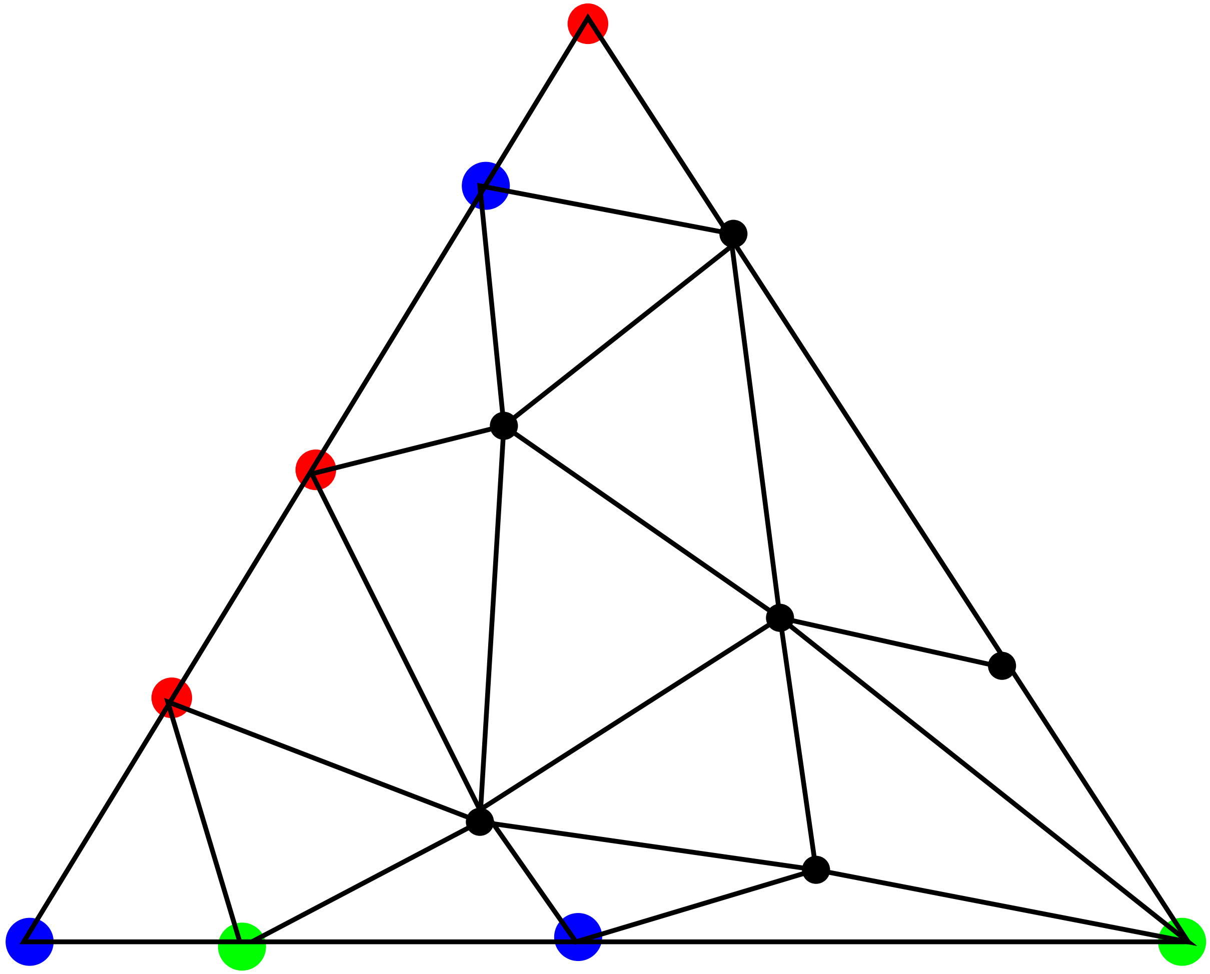
Let T be a triangulation of the n-dimensional simplex. Suppose the points of T are coloured with a Sperner colouring.

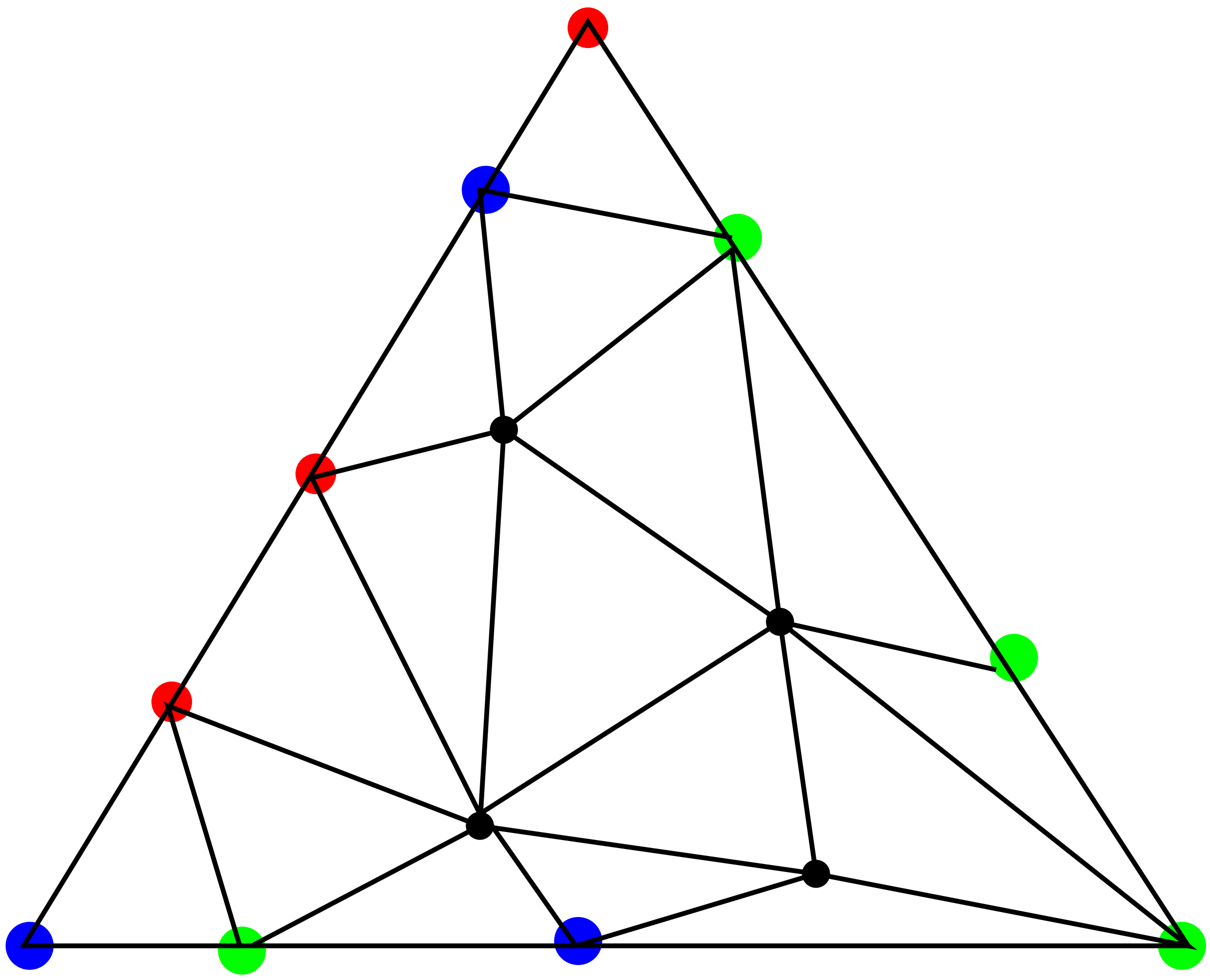
Then there exists a multicoloured elementary simplex in this colouring.

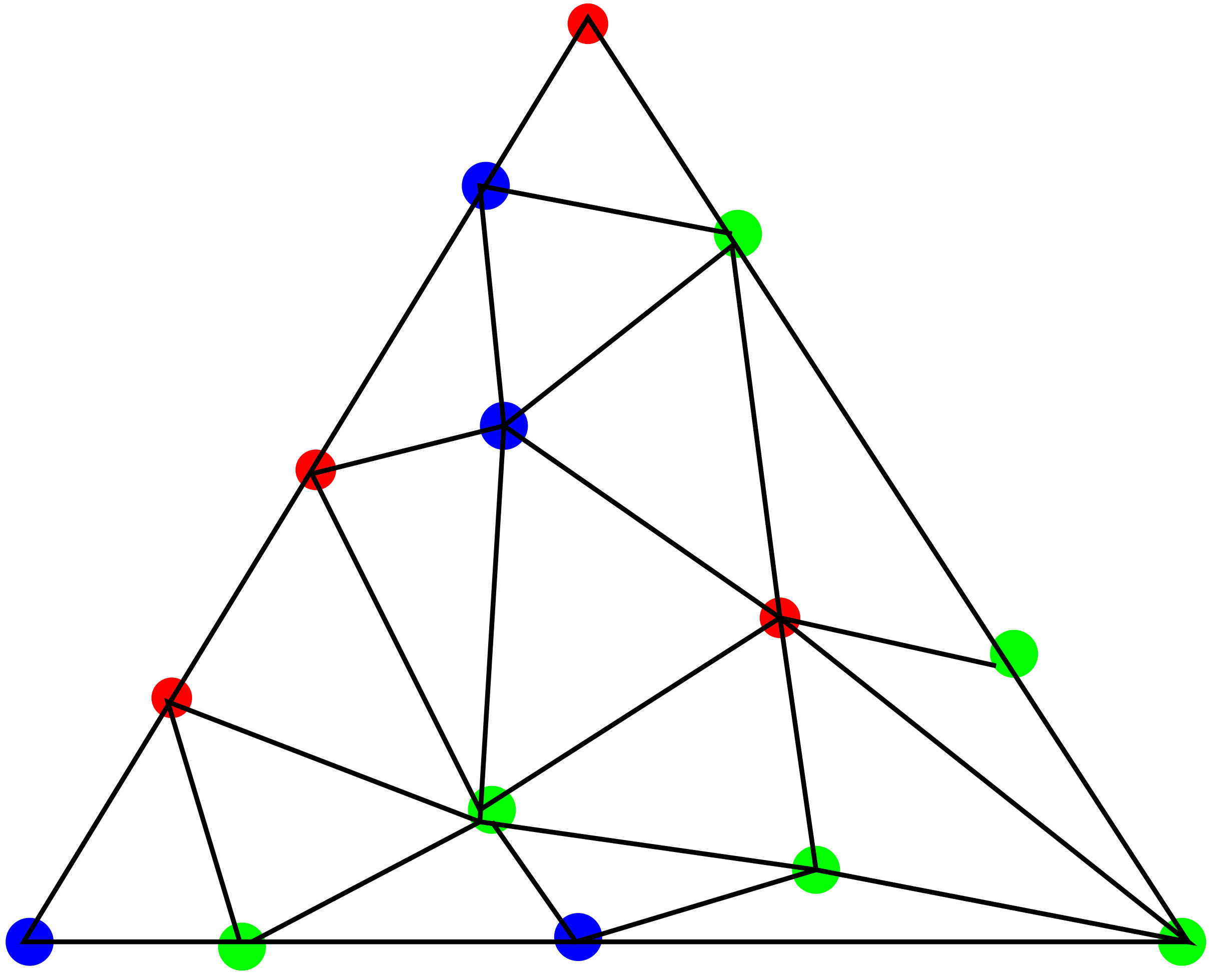


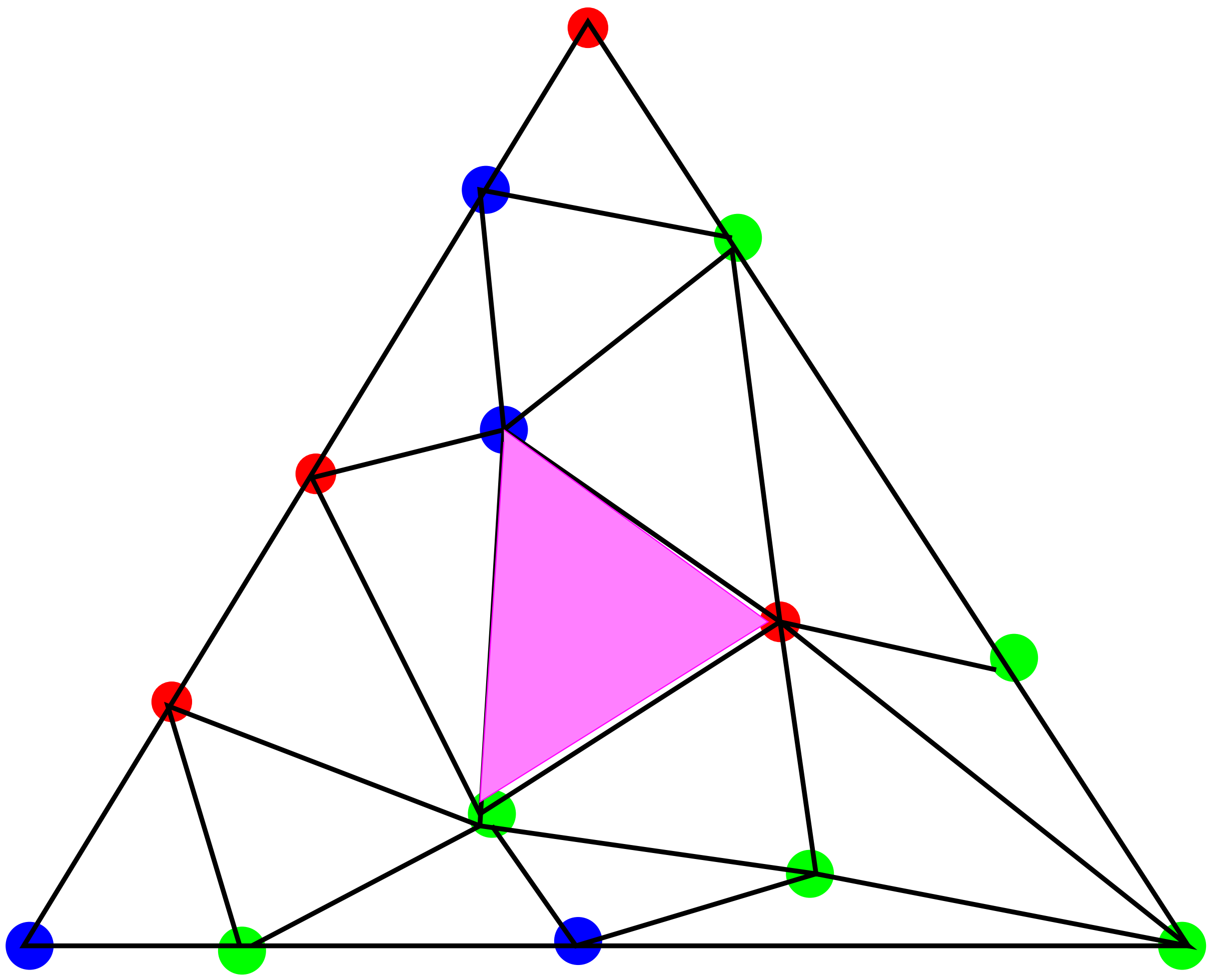












Q: Can we transfer the problem of finding a multicoloured simplex in the independence complex of a graph to that of finding a multicoloured simplex in a triangulation of the simplex?

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How?

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A: YES!

How?

Using topological connectedness.

How to find an independent transversal?

Let G be a graph with vertex partition $U_1, \dots, U_m$. Suppose we can construct a triangulation T of the $(m-1)$-dimensional simplex, and a simplicial map f from T to $\mathcal{I}(G)$ such that the induced colouring on T is a Sperner colouring.

A simplicial map f from a simplicial complex Δ to a simplicial complex Σ is a function $f : V(\Delta) \to V(\Sigma)$ such that

$f(\tau) = \{f(w) : w \in V(\tau)\}$ is a simplex of Σ for each simplex τ of Δ.

Then T contains a multicoloured simplex, which gives an independent transversal in G.

Q: What conditions will guarantee such a simplicial map exists?

Topological connectedness

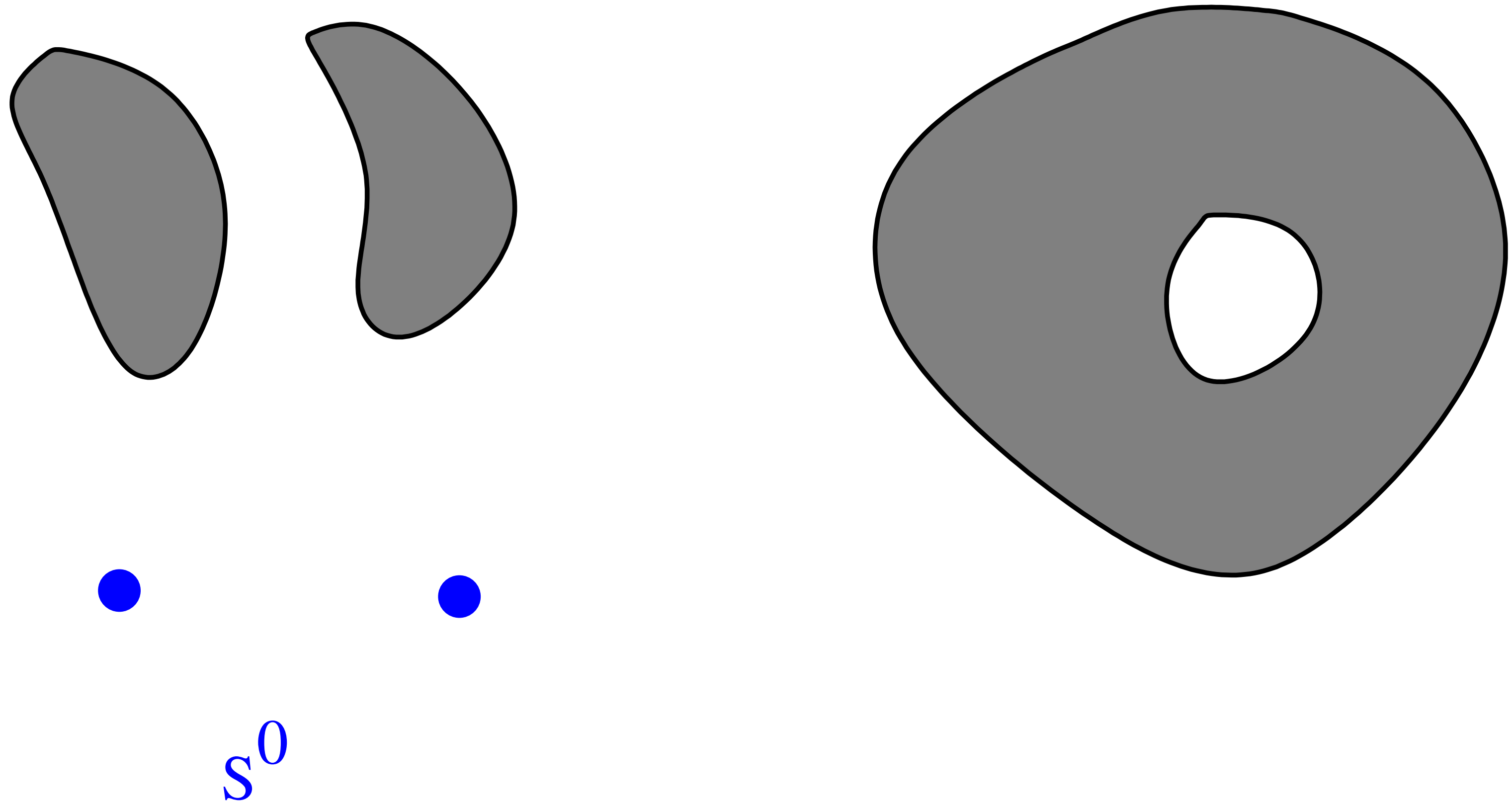
A topological space X is said to be k-connected if

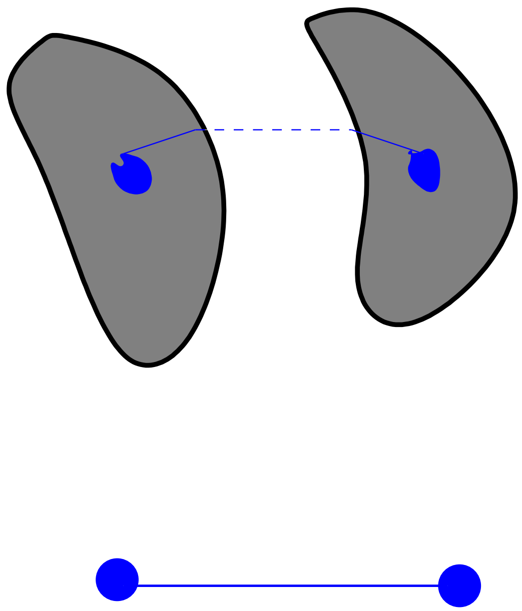
- for each $-1 \leq d \leq k$, and
- for each continuous map f from the d-sphere S^d to X

there exists a continuous map f' from the $(d+1)$-ball B^{d+1} to X that extends f.

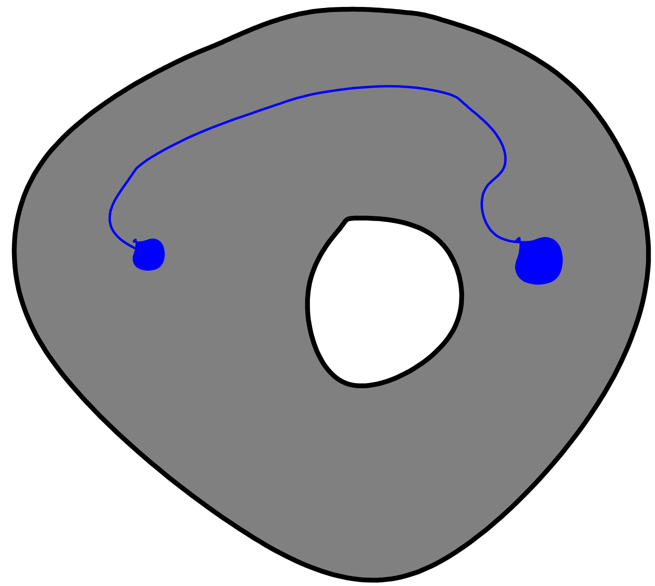
i.e. "there are no holes up to dimension k".

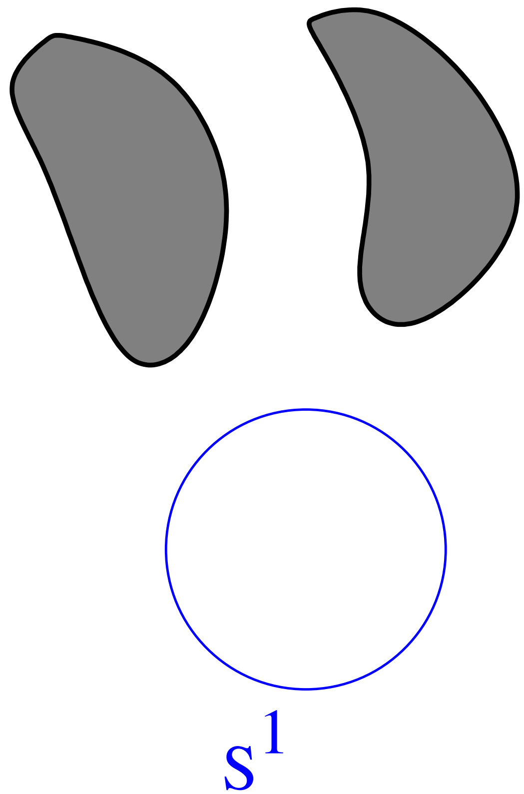
In particular -1-connected means nonempty.

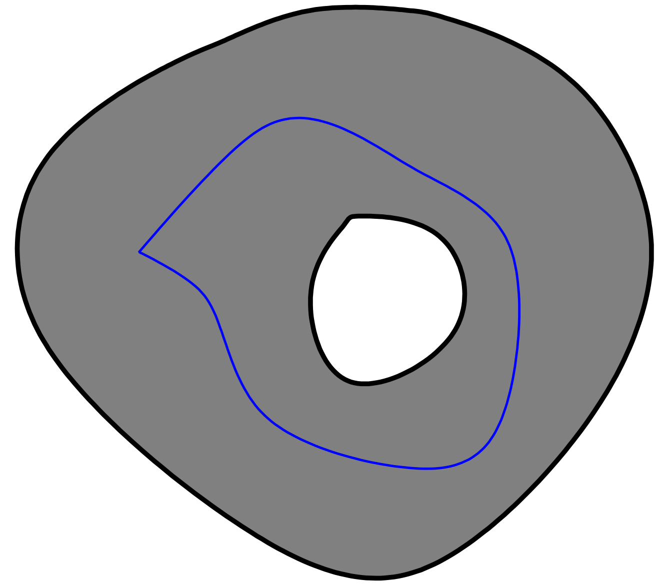

S^0



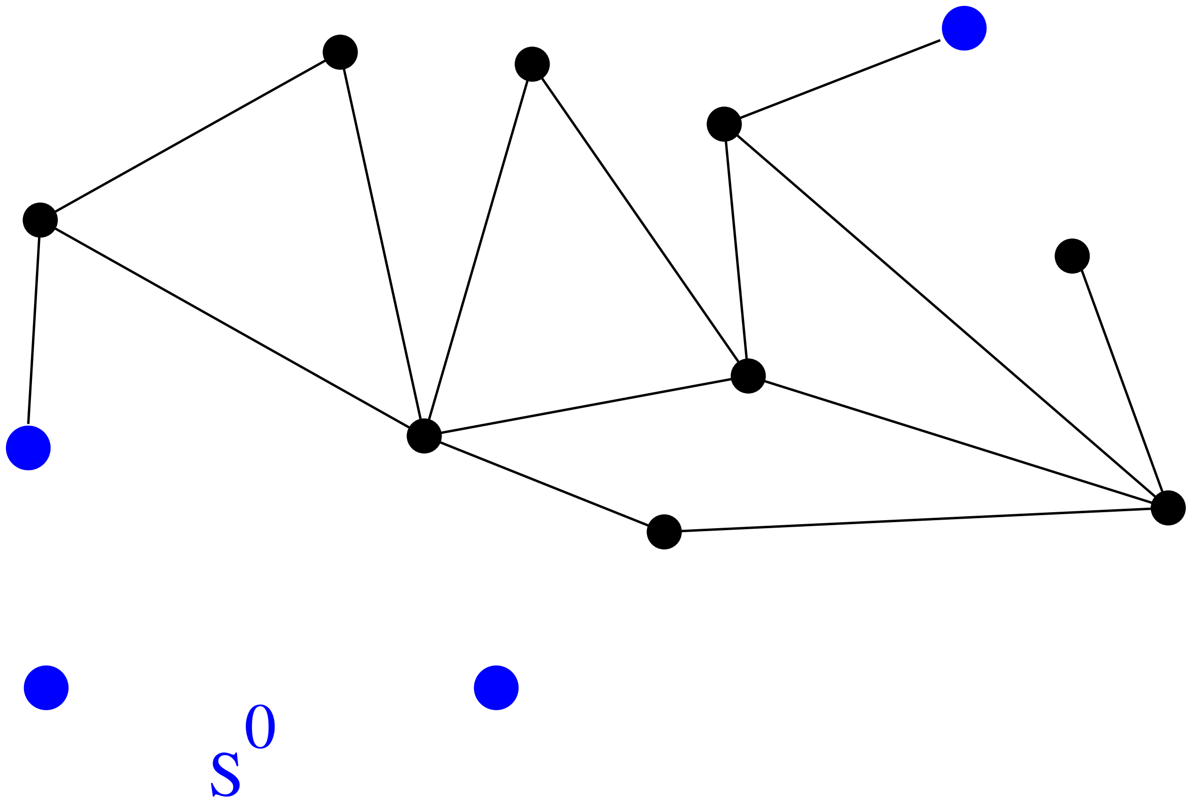
B^1



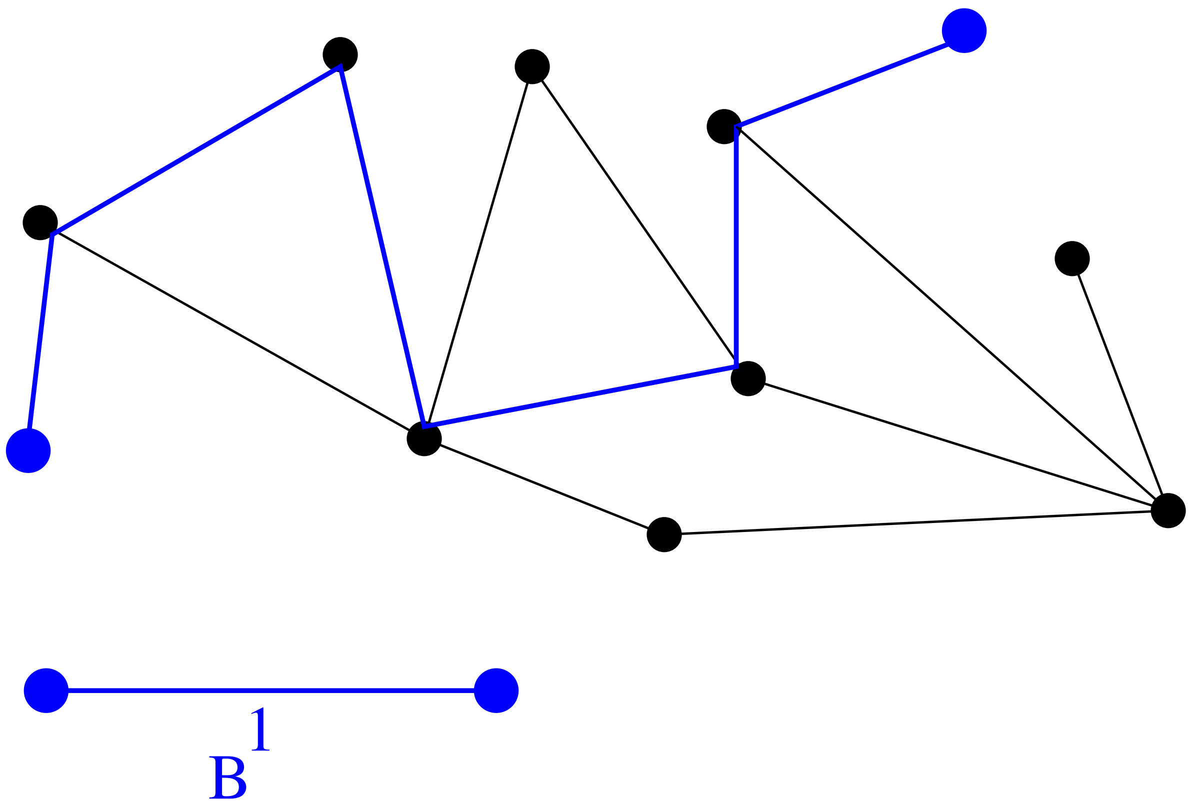

S^1



Connectedness vs connectivity



Topological 0-connectedness corresponds to connected in a graph.


B^1

Connectedness

If the simplicial complex Σ is k-connected then

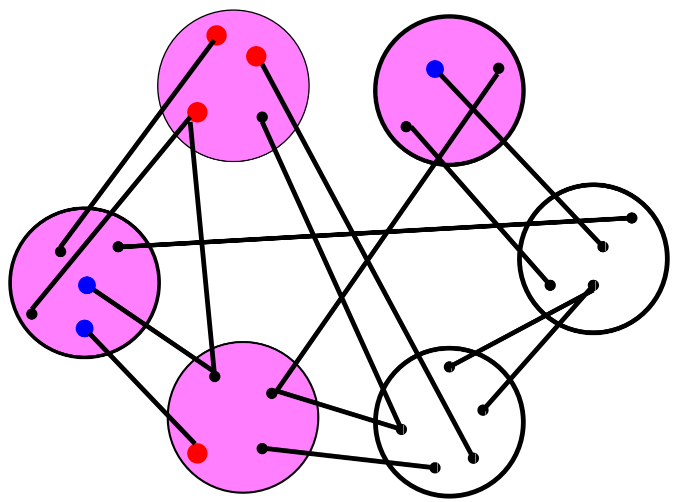
- for each $-1 \leq d \leq k$,
- for each triangulation T of the boundary of a $(d+1)$-simplex, and
- for each simplicial map f from T to Σ,

the triangulation T can be extended to a triangulation T' of the whole $(d+1)$-simplex, and f can be extended to a simplicial map f' from T' to Σ.

View Σ as a topological space via its geometric realization, OR just use the above as definition.

Independent Transversals

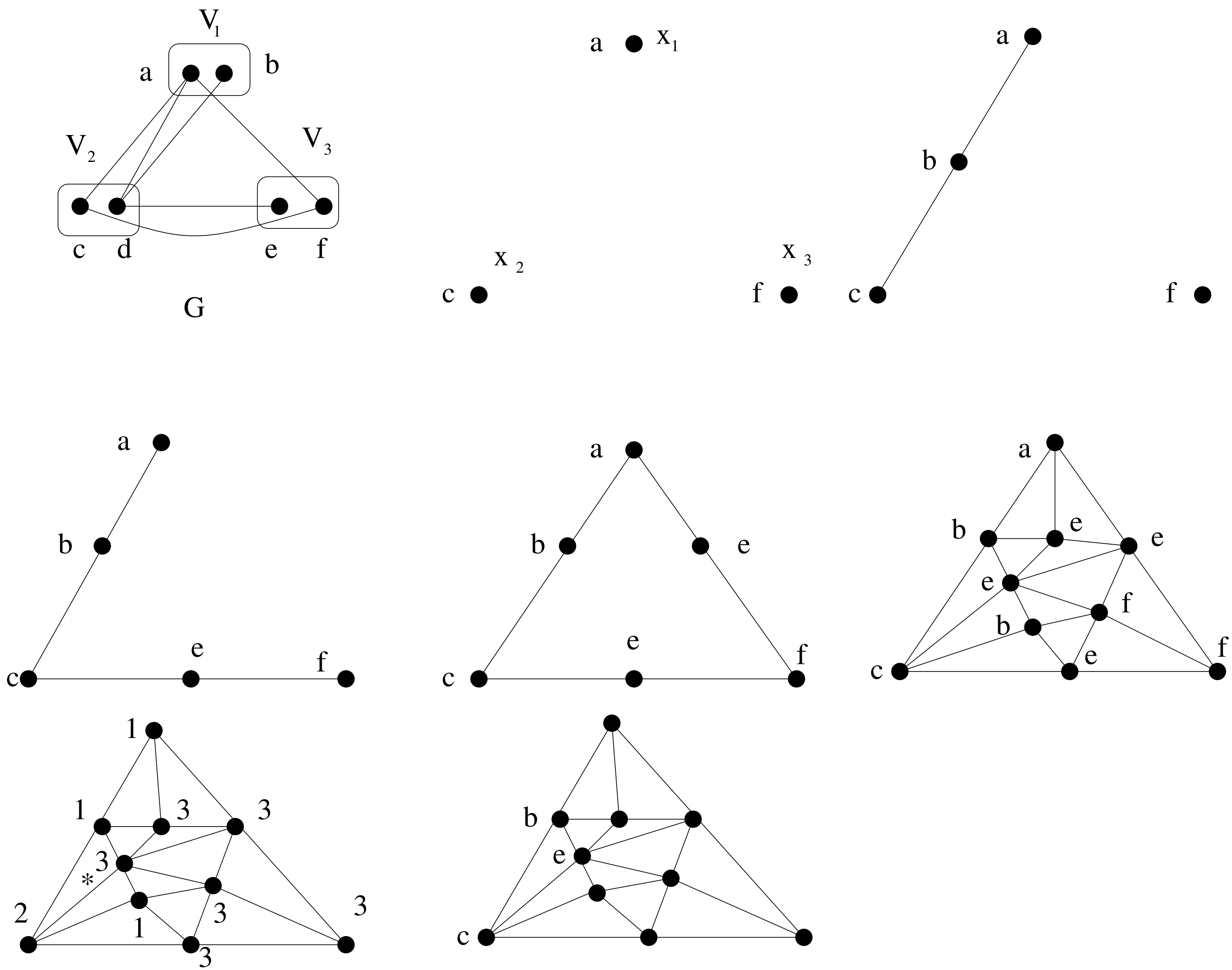
THEOREM:(Aharoni-H, Aharoni-Berger) Let G be a graph with vertex partition $U_1, \ldots, U_m$. Suppose that for each $S \subseteq [m]$, the subcomplex $\mathcal{I}(G_S)$ of independent sets in $G_S = G[\bigcup_{i \in S} U_i]$ is $(|S| - 2)$-connected.



Then G has an independent transversal with respect to the vertex partition $U_1, \ldots, U_m$.

Proof:

Build up a suitable triangulation T of the $(m-1)$-dimensional simplex, and a suitable simplicial map from T to $\mathcal{I}(G)$, starting with the 0-dimensional faces and proceeding face by face in order of dimension.


V1
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Then theory about topological connectedness helps obtain lower bounds on the connectedness of the $\mathcal{I}(G_S)$, if we know certain special properties about the graph G and its vertex partition.

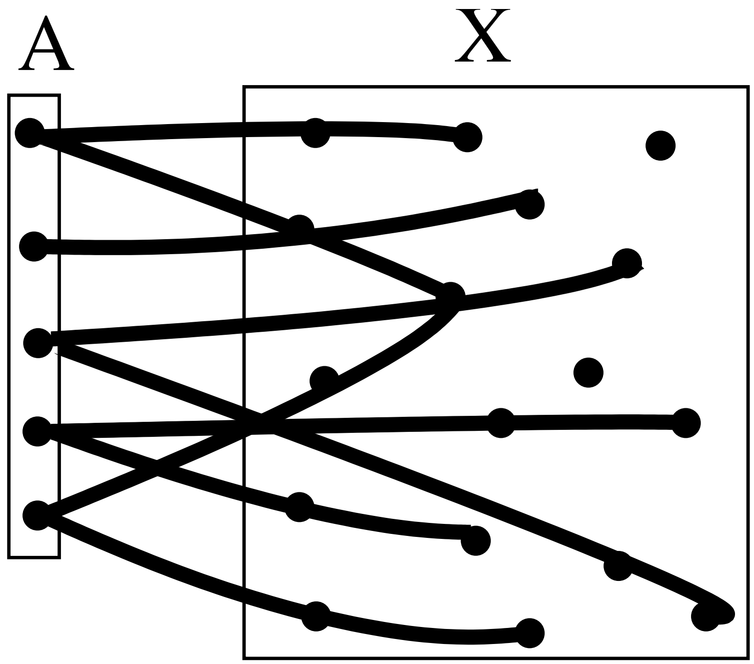
Q: What properties of a graph influence the topolgical connectedness of its independence complex?

Several properties involving domination in G turn out to be related to the connectedness of $\mathcal{I}(G_S)$.

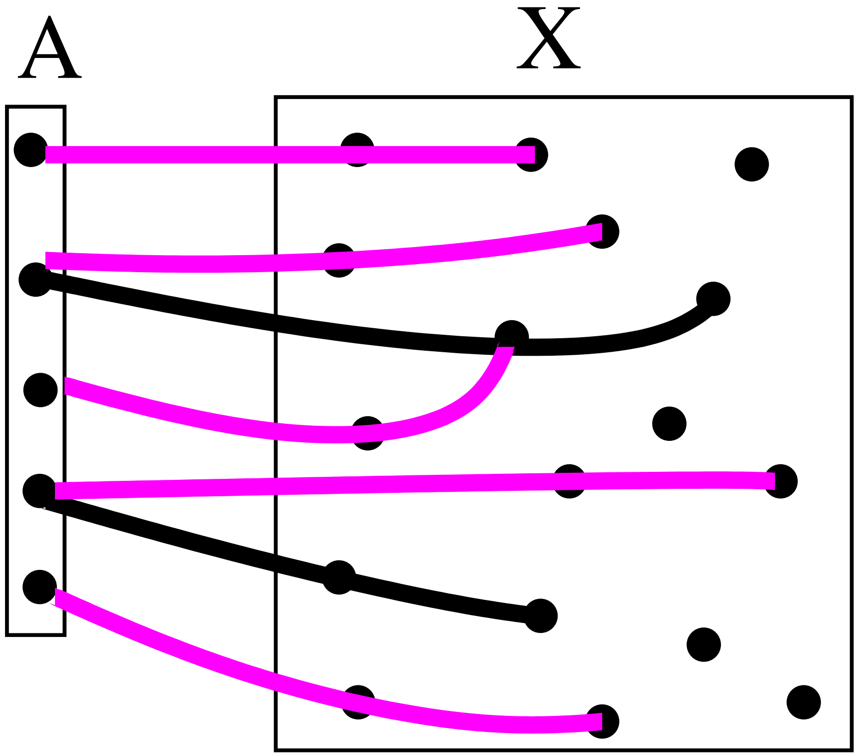
Here we will focus on the case in which G is the line graph of a hypergraph, to obtain results on hypergraph matching.

Matching in bipartite hypergraphs

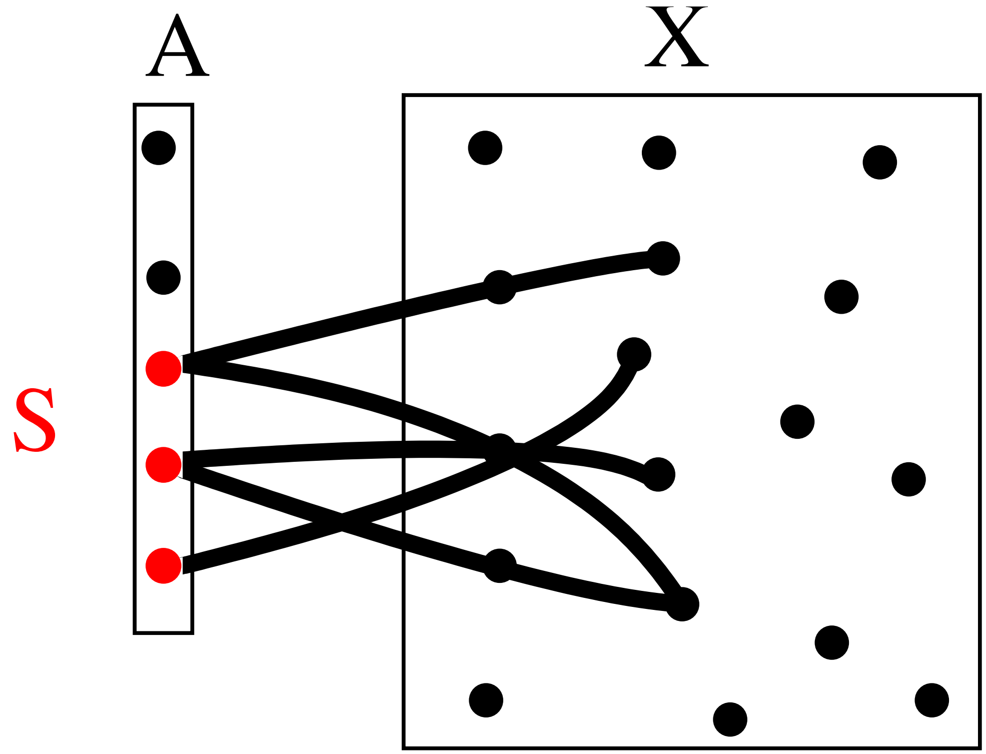
def: A bipartite 3-uniform hypergraph:

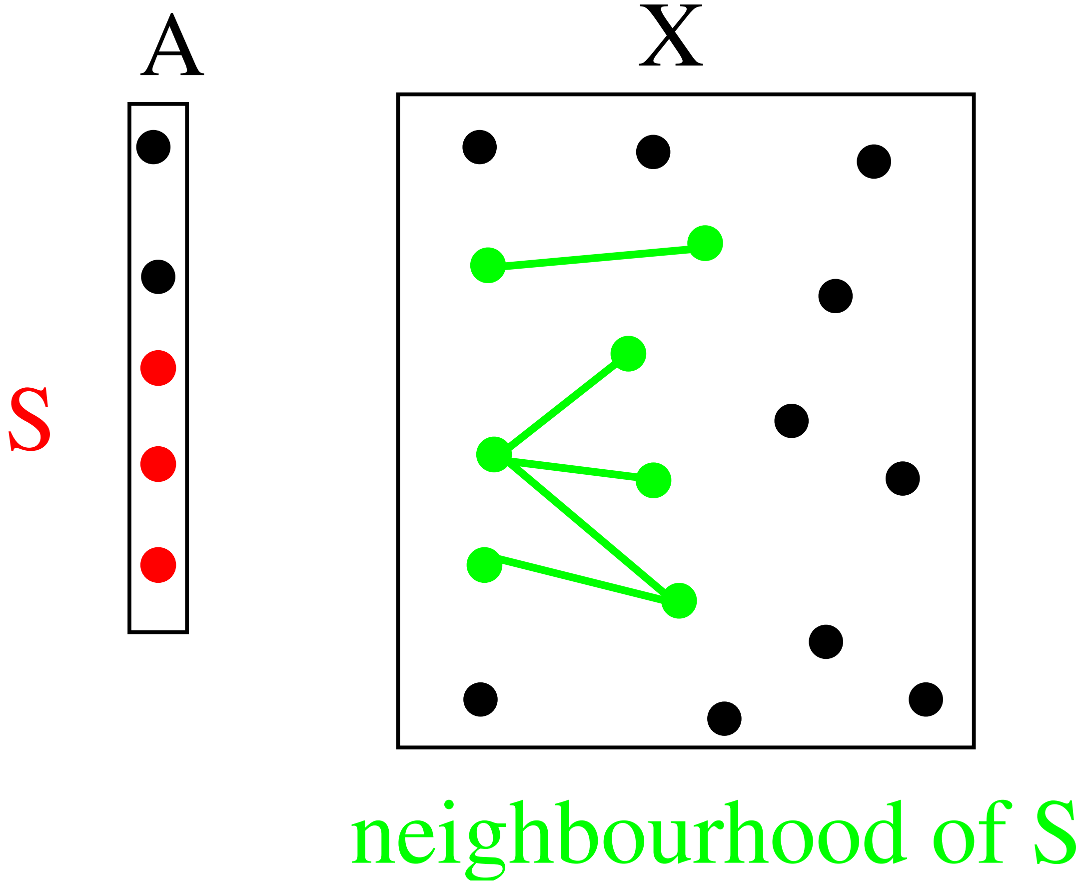


def: A complete hypermatching:



def: The neighbourhood (link) $\Gamma(S)$ of a subset S of A:




A
X
S
neighbourhood of S

Formulated in terms of independent transversals

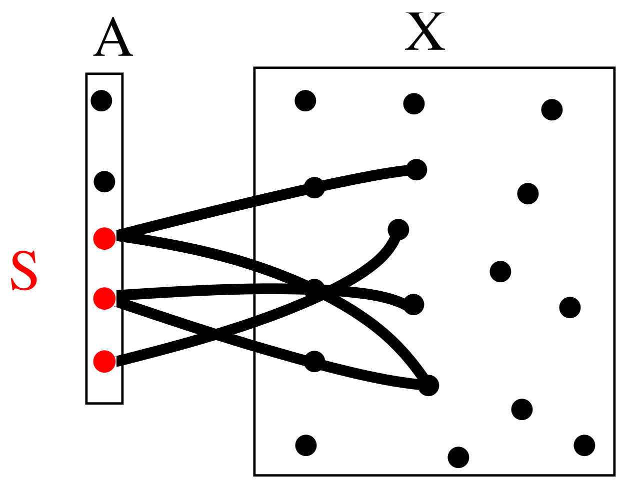
Given a bipartite hypergraph $\mathcal{J}$ with vertex classes A and X, define the graph G to be the line graph of $\Gamma(A)$.

The partition classes are determined by the elements of A: we put $e \in V(G)$ into the class corresponding to $a \in A$ precisely when $e \cup \{a\} \in \mathcal{J}$.

Then the independent transversals in G correspond precisely to the complete hypergraph matchings in $\mathcal{J}$.

Hall's Theorem for r-uniform hypergraphs

THEOREM:(Aharoni-H) The bipartite r-uniform hypergraph J has a complete matching if: For every subset $S \subseteq A$, the $(r-1)$-uniform neighbourhood hypergraph $\Gamma(S)$ has a matching that has size at least $(r-1)(|S|-1)+1$.



Hall's Theorem for hypergraphs follows from

Fact. If a graph G contains an independent set that is not totally dominated by a set of at most $t+1$ vertices of G, then $\mathcal{I}(G)$ is t-connected.

This Fact is applied to the partitioned line graph G of $\Gamma(A)$. In particular if a hypergraph contains a large matching then it is hard to dominate.

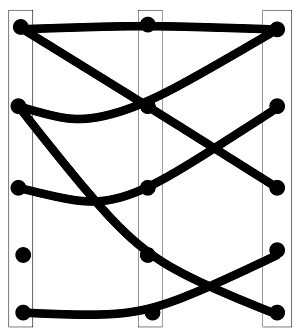
Ryser's Conjecture

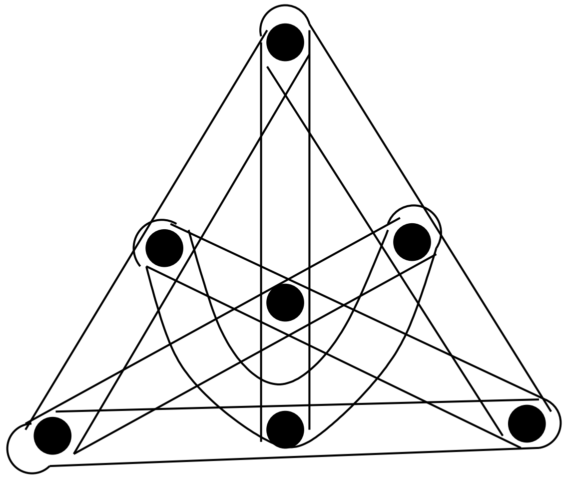
A cover of the hypergraph $\mathcal{H}$ is a set of vertices C of $\mathcal{H}$ such that every edge of $\mathcal{H}$ contains a vertex of C. The parameter $\tau(\mathcal{H})$ is defined to be the minimum size of a cover of $\mathcal{H}$. We denote by $\nu(\mathcal{H})$ the maximum size of a matching in $\mathcal{H}$.

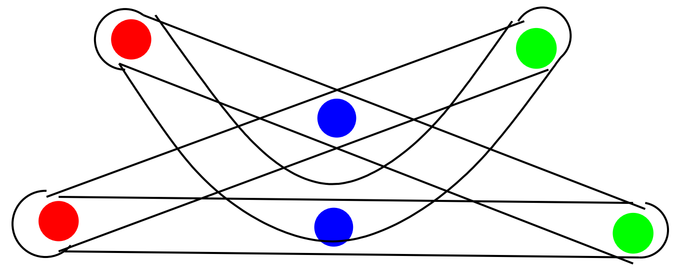
Note that for every r-uniform hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$.

Ryser's Conjecture: Let $\mathcal{H}$ be an r-partite r-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H}).$$







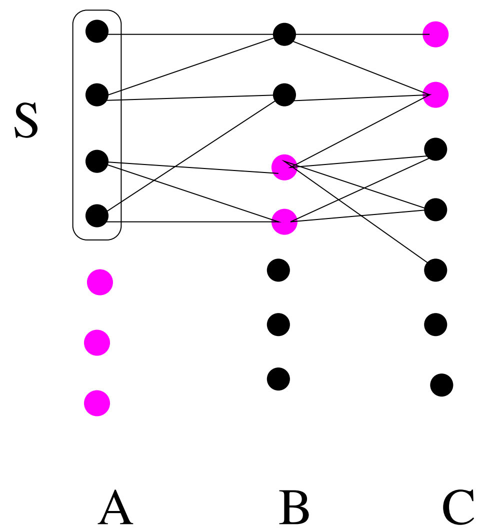
Ryser's Conjecture

THEOREM (Aharoni 2001): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

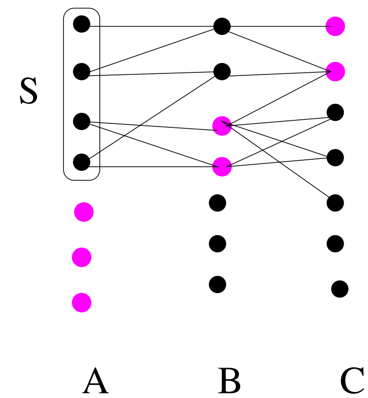
The proof uses a defect version of Hall's Theorem for Hypergraphs.

Proof Idea



Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Then every subset S of A gives a cover of $\mathcal{H}$ of size $|A| - |S| + \tau(G_S)$.

Here G_S is the (bipartite) multigraph of pairs bc such that $abc \in \mathcal{H}$ for some $a \in S$.

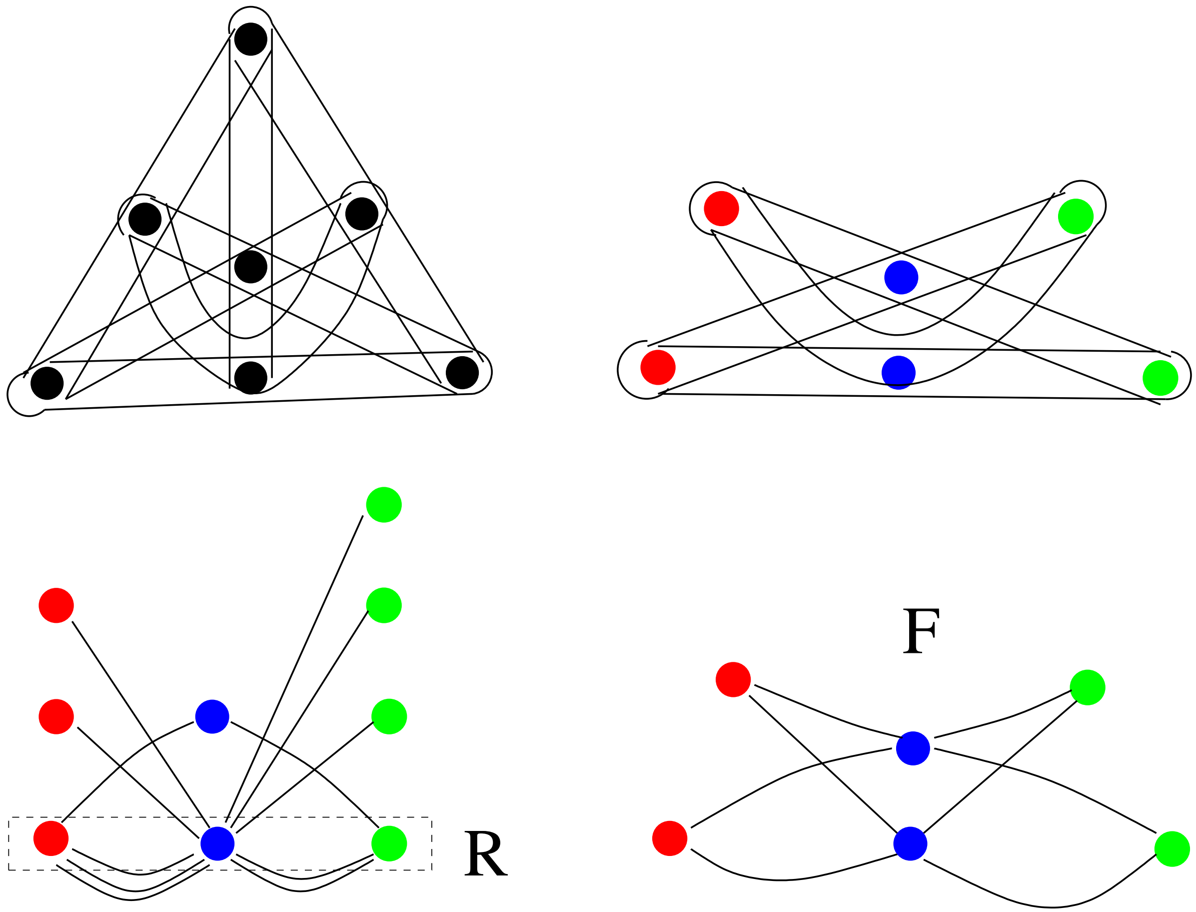


By König's Theorem $\tau(G_S) = \nu(G_S)$, so

$$\tau(\mathcal{H}) \leq |A| - |S| + \nu(G_S).$$

Therefore for every $S \subseteq A$ we find

$$\nu(G_S) \geq |S| - |A| + \tau(\mathcal{H}).$$


F
R

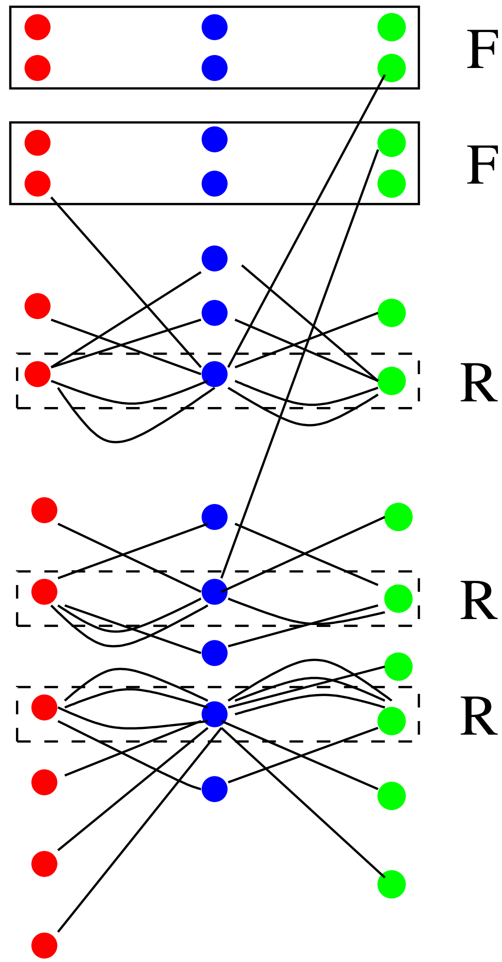
Extremal hypergraphs for Ryser's Conjecture

THEOREM (H, Narins, Szabó 2018): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Suppose

$$\tau(\mathcal{H}) = 2\nu(\mathcal{H}).$$

Then $\mathcal{H}$ is a home base hypergraph.

The proof involves a characterisation of bipartite graphs for which the connectedness of the independence complex of their line graphs is as small as possible with respect to their matching number.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R

Stability for matchings in regular hypergraphs

THEOREM: Let $r > 0$ be given. Every r-regular 3-partite 3-uniform (multi)hypergraph, with n vertices in each class, has a matching of size at least $n/2$.

This is easily implied by Aharoni's Theorem. It is best possible for all even r and all even n: for an example take $n/2$ disjoint copies of $\frac{r}{2} \cdot F$ ($(r/2)$ multiples of the hypergraph F).

THEOREM (H, Narins 2018): Let $\mathcal{H}$ be an r-regular 3-partite 3-uniform (multi)hypergraph with n vertices in each class, with $\nu(\mathcal{H}) \leq (1+\varepsilon)\frac{n}{2}$. Then $\mathcal{H}$ has at least

$$\left(1 - \left(22r - \frac{77}{3}\right)\varepsilon\right)\frac{n}{2}$$

components that are copies of $\frac{r}{2} \cdot F$.

Open Problems

- The last theorem implies that if the r-regular 3-partite 3-uniform (multi)hypergraph with n vertices in each class contains no copy of $\frac{r}{2} \cdot F$ then

$$\nu(\mathcal{H}) \geq \left(1 + \frac{1}{22r - \frac{77}{3}}\right) \frac{n}{2}.$$

  This is close to being best possible, since examples exist with $\nu(\mathcal{H}) \leq \left(1 + \frac{1}{r}\right) \frac{n}{2}$. It is natural to conjecture that this is the right value.

- It is believed that much stronger bounds should hold if $\mathcal{H}$ is simple, i.e. does not have multiple edges. There exist simple r-regular 3-partite 3-uniform hypergraphs $\mathcal{H}$ for which $\nu(\mathcal{H}) = \frac{2n}{3}$, and this could be the correct bound. A lower bound of $\frac{3}{5}$ for the $r = 3$ case was proved by Cavenaugh, Kuhl and Wanless.

- More generally, Alon and Kim conjectured that the edges of every simple 3-uniform hypergraph with maximum degree r can be partitioned into $(\frac{3}{2} + o(1))r$ matchings.

- No stability result for Ryser's Conjecture for 3-partite 3-uniform hypergraphs is currently known.

- Ryser's Conjecture is still wide open for all $r \geq 4$. For $r \geq 6$ no nontrivial bound is known.