Near-$(3k-4)$-Leray property of the complex of graphs with no matchings of size k

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Overview

Outline for section 1

A theorem on the existence of a rainbow matchings

In this presentation, we identify a graph G with its edge set $E(G)$.

A theorem on the existence of a rainbow matchings

In this presentation, we identify a graph G with its edge set $E(G)$.

Theorem

Let $E_1, \ldots, E_{3k-2}$ be sets of edges in an arbitrary graph such that

$$\nu(E_i \cup E_j) \geq k \text{ for every } i \neq j.$$

Then, there exists a rainbow matching of size k.

- In [ABCHS, 2018], the existence of a rainbow matching of size k is proved for E_i satisfying $\nu(E_i) \geq k$ for every $1 \leq i \leq 3k-2$.
- In [AHJ, 2018], analogue results on fractional matchings are proved in hypergraph setting.

Topological theorems

A simplicial complex K is called near-d-Leray if

$$\tilde{H}_i(lk_K(\sigma)) = 0 \text{ for every } \emptyset \neq \sigma \in K \text{ and } i \geq d.$$

Theorem

$NM_k(H) := \{G \subseteq H : \nu(G) < k\}$ is near-$(3k-4)$-Leray.

Theorem (Holmsen, 2016)

Let K be a **near-d-Leray** *simplicial complex, and (M, ρ) be a matroidal complex* **of dimension $> d$** *on the same V. If $M \leq K$, then there exists $\sigma \in K$ such that $\rho(V \setminus \sigma) \leq d$.*

If the *near-d-Leray* condition is changed to *d-Leray* and the dimension condition on M is deleted, then it becomes the theorem of Kalai and Meshulam [KM, 2005].

Proof of the existence of the rainbow matching

Theorem (Holmsen, 2016)

Let K be a near-d-Leray simplicial complex, and (M, ρ) be a matroidal complex of dimension $> d$ on the same V. If $M \leq K$, then there exists $\sigma \in K$ such that $\rho(V \setminus \sigma) \leq d$.

- $\tilde{E}_i := \{(e, i) : e \in E_i\}$, $\tilde{E} := \bigcup_{i \in [3k-2]} \tilde{E}_i$, and
- $K := \{\tilde{E}' \subseteq \tilde{E} : \nu(\{e : \exists i \; s.t. (e, i) \in \tilde{E}'\}) < k\}$.

Proof sketch.

Let $d = 3k - 4$ and let M be a partition matroid given by the $\tilde{E}_i$ ($\Rightarrow \dim M = (3k-2) - 1 = 3k - 3 > d$). Suppose the contrary. Then $M \leq K$. Since $NM_k(\bigcup_{i=1}^{3k-2} E_i)$ is near-d-Leray, K is also near-d-Leray ($\because lk_K(\sigma)$ for $\emptyset \neq \sigma \in K$ is a star or $\simeq$ a link in $NM_k(\bigcup E_i)$). By Holmsen's theorem, there is an edge set $\sigma \in K$ ($\nu(\sigma) < k$) which contains some $\tilde{E}_i \cup \tilde{E}_j$. $\Rightarrow\Leftarrow$ $\square$

Hereditary property of the near-d-Leray property

Theorem (Restatement)

$NM_k(H) := \{G \subseteq H : \nu(G) < k\}$ *is near-*$(3k-4)$*-Leray.*

- Near-d-Leray property is hereditary: If L is an induced subcomplex of K and K is near-d-Leray, then L is also near-d-Leray.

Theorem

$NM_k(K_n)$ *is near-*$(3k-4)$*-Leray.*

- In [LSW, 2004], it is shown that $NM_k(K_n)$ is homotopy equivalent to a wedge of $(3k-4)$-spheres using the discrete Morse theory. We follow and generalize their proof arguments.

Outline for section 2

1. Connection of the near-d-Lerayness to the existence of a rainbow matching
2. Tools
   - The discrete Morse theory
   - The Gallai-Edmonds decomposition
3. Proof of the main theorem
4. Some questions

The main theorem of Discrete Morse theory

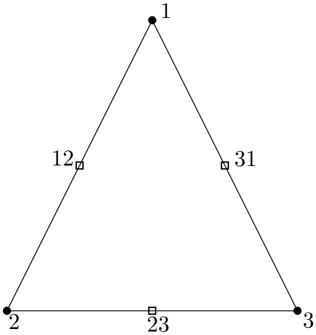
Let K be a simplicial complex, and $\mathcal{P} = \mathcal{F}(K)$.

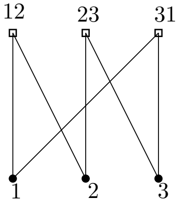
- $\mathcal{D}(\mathcal{P})$ is the directed graph obtained by directing each edge in the Hasse diagram of $\mathcal{P}$ downwards.
- For a matching $\mathcal{M}$ in $\mathcal{D}(\mathcal{P})$, let $\mathcal{D}_{\mathcal{M}}(\mathcal{P})$ be the directed graph obtained by reversing direction of each edge of $\mathcal{M}$ in $\mathcal{D}(\mathcal{P})$. If $\mathcal{D}_{\mathcal{M}}(\mathcal{P})$ does not contain a directed cycle, then $\mathcal{M}$ is called a **Morse matching** of $\mathcal{P}$.
- Faces of K not covered by $\mathcal{M}$ are called **critical cells**.

Theorem (Forman)

If there is a Morse matching of a face poset of K with c_i i-dim critical cells for each i, then $K \simeq K'$ where K' is a CW-complex with c_i i-dim faces for each i.

An example





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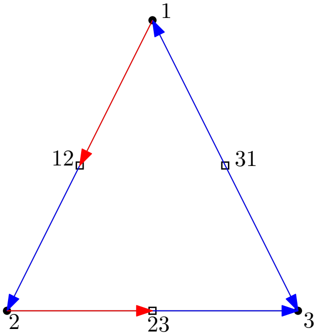
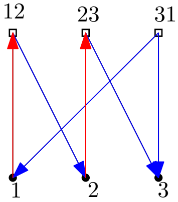
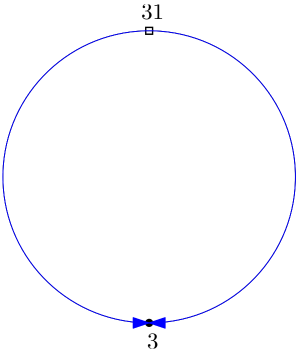
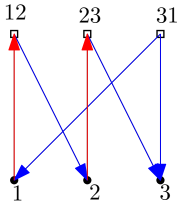




An example





Upper bound on the size of critical cells and the near-d-Leray property

Theorem

Given a subgraph H of K_n with $\nu(H) < k$, the poset $\{G \subseteq K_n : H \subseteq G, \nu(G) < k\}$ has a Morse matching whose critical cells have size $\leq 3k - 4 + |H|$.

Theorem (Restatement)

For every nonempty face H of $NM_k(K_n)$,

$$\tilde{H}_i(lk_{NM_k(K_n)}(H)) = 0 \text{ for every } i \geq 3k-4.$$

Here, $NM_k(K_n) := \{G \subseteq K_n : \nu(G) < k\}$.

Proof.

$lk_{NM_k(K_n)}(H)$ has a Morse matching whose critical cells have size $\leq 3k-4$ ($\Rightarrow$ dimension $\leq 3k-5$). □

The Gallai-Edmonds decomposition

For $G = (V, E)$, the Gallai-Edmonds decomposition of G is

- $D = D(G) := \{v \in V : \exists \text{ maximum matching of } G \text{ not covering } v\}$,
- $A = A(G) := \{v \in V \setminus D : N_G(v) \cap D \neq \emptyset\}$, and
- $C = C(G) := V \setminus (A \cup D)$.

Theorem (Gallai-Edmonds structure theorem)

1. *Every odd component H of $G - A$ is factor-critical (i.e. $\forall v \in V(H)$ $H - v$ has a perfect matching) and $V(H) \subseteq D$.*
2. *Every even component H of $G - A$ has a perfect matching and $V(H) \subseteq C$.*

The Gallai-Edmonds decomposition

For $G = (V, E)$, the Gallai-Edmonds decomposition of G is

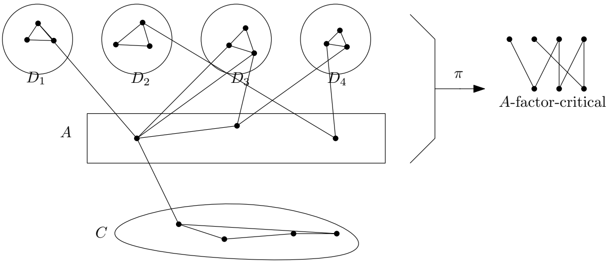
- $D = D(G) := \{v \in V :$ $\exists$ maximum matching of G not covering $v\}$,
- $A = A(G) := \{v \in V \setminus D : N_G(v) \cap D \neq \emptyset\}$, and
- $C = C(G) := V \setminus (A \cup D)$.

Theorem (Gallai-Edmonds structure theorem, continued)

3. *$\forall A' \subseteq A$, $N_G(A')$ intersects $> |A'|$ components of $G[D]$.*
4. *If $con(G)$ is the number of connected components of $G[D]$, then*

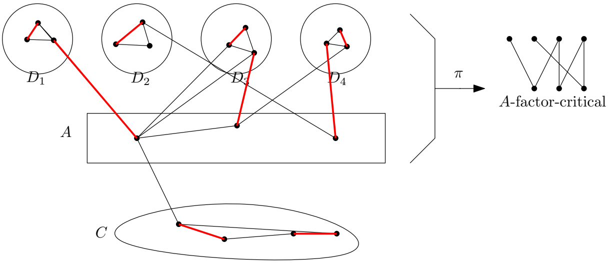
$$con(G) = |A| + |V| - 2\nu(G).$$

The Gallai-Edmonds decomposition



- A bipartite graph G on $\tilde{D} \cup A$ is ***A*-factor-critical** if for every $v \in \tilde{D}$, $G - v$ has a matching covering A.
- $\forall A' \subseteq A(G)$, $N(A')$ intersects $> |A'|$ components of $G[D]$. $\Leftrightarrow \pi(G[D, A])$ is A-factor-critical. ($G[D, A]$ is the set of edges in G between D and A.)

The Gallai-Edmonds decomposition



A maximum matching uses all vertices in $A \cup C$ but misses some in $D = \bigcup D_i$.

Outline for section 3

Step 0: Constructing Morse matchings for each missing edge of H incident with v_n

- Find a vertex v_n such that there exists a missing edge of H **incident with** v_n, and an edge of H **not incident with** v_n (assume $N_H(v_n) = \{v_{m+1}, v_{m+2}, \ldots, v_{n-1}\}$).
- While j ranging from 1 to m, define

$$\mathcal{M}^{(j)} := \{(G, G - v_j v_n) : v_j v_n \in G, G \in \mathcal{C}^{(j-1)}\},$$

where $\mathcal{C}^{(0)} := \{G \subseteq K_n : H \subseteq G, \nu(G) < k\}$ and $\mathcal{C}^{(j)}$ is the subposet of graphs which are not covered by earlier matchings $\mathcal{M}^{(j)}$.

- $\overline{\mathcal{M}^{(j)}}$ is the poset of graphs covered by $\mathcal{M}^{(j)}$.
- For $G_1 \in \overline{\mathcal{M}^{(j)}}$ and $G_2 \in \mathcal{C}^{(j)}$, either $G_1 \subseteq G_2$, or G_1 and G_2 are not comparable. $\Rightarrow$ One can define a partial order in $\{\overline{\mathcal{M}^{(j)}}, \mathcal{C}^{(j)}\}$.

Step 0: Using the cluster lemma and the cycle lemma

Lemma (Cluster lemma)

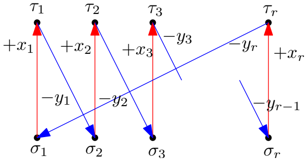
Let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be pairwise disjoint, order convex subposets of $\mathcal{P}$. For each $i \in [r]$, let $\mathcal{M}_i$ be a Morse matching on $\mathcal{P}_i$. Define a relation on the $\mathcal{P}_i$ by $\mathcal{P}_i \leq_c \mathcal{P}_j$ if there exist $x \in \mathcal{P}_i$ and $y \in \mathcal{P}_j$ such that $x \leq y$. If $\leq_c$ is a partial order, then $\bigcup_{i \in [r]} \mathcal{M}_i$ is a Morse matching on $\mathcal{P}$.

Step 0: Using the cluster lemma and the cycle lemma

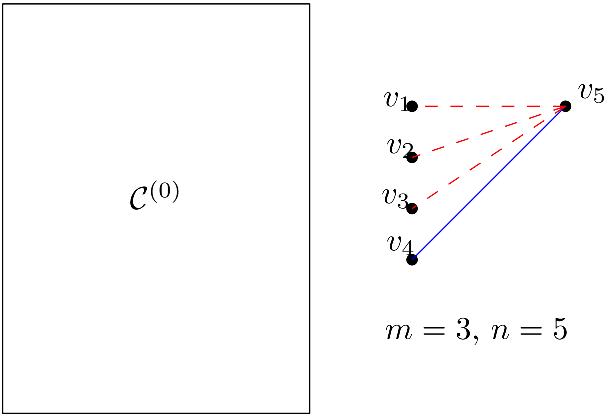
Lemma (Cycle lemma)

Let $\mathcal{P}$ be an order convex subposet of $\mathcal{F}(K)$ where K is a simplicial complex, and $\mathcal{M}$ be a matching of $\mathcal{D}(\mathcal{P})$. Then every directed cycle in $\mathcal{D}_{\mathcal{M}}(\mathcal{P})$ is of the form $\sigma_1, \tau_1, \dots, \sigma_r, \tau_r, \sigma_{r+1} = \sigma_1$ for some $r \geq 2$, where

1. $\forall i \in [r]\ \exists x_i \in \tau_i$ *s.t.* $\tau_i = \sigma_i \cup \{x_i\}$ *and* $(\tau_i, \sigma_i) \in \mathcal{M}$.
2. $\forall i \in [r]\ \exists y_i \in \tau_i$ *s.t.* $\tau_i = \sigma_{i+1} \cup \{y_i\}$ *and* $(\tau_i, \sigma_i) \in \mathcal{M}$.
3. $\{x_i : i \in [r]\} = \{y_i : i \in [r]\}$ *as multisets.*

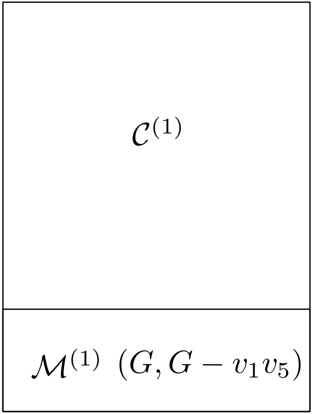


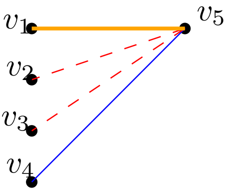
Step 0: Using the cluster lemma and the cycle lemma



$m = 3,\ n = 5$

Step 0: Using the cluster lemma and the cycle lemma





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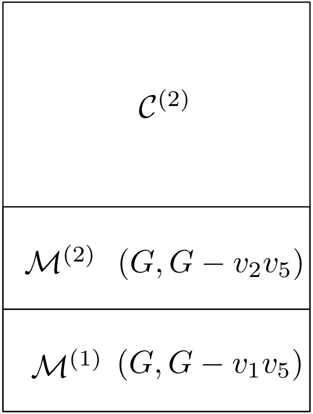
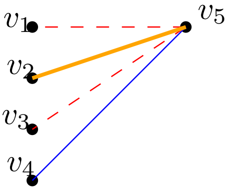
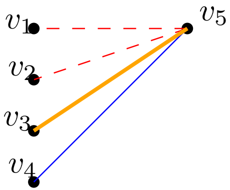




$m = 3,\ n = 5$

Step 0: Using the cluster lemma and the cycle lemma

$\mathcal{C}^{(3)}$
$\mathcal{M}^{(3)}$ $(G, G - v_3v_5)$
$\mathcal{M}^{(2)}$ $(G, G - v_2v_5)$
$\mathcal{M}^{(1)}$ $(G, G - v_1v_5)$



$m = 3,\ n = 5$

Step 1: Partitioning into possible Gallai-Edmonds decompositions $(D_1, \dots, D_c; A; C)$

The remaining graphs in $\mathcal{C}^{(m)}$ are exactly graphs G such that for every $j \in [m]$

1. $v_j v_n \notin G$, and
2. $\nu(G + v_j v_n) = k$ $(\Rightarrow \nu(G - v_n) = k - 1)$.

A Gallai-Edmonds decomposition of $G - v_n$ is of the form $(D = \{v_1, \dots, v_m\}, A, C)$.

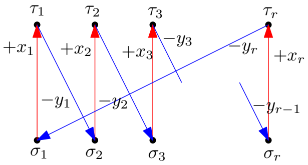
We partition $\mathcal{C}^{(m)}$ into subposets of graphs sharing the same Gallai-Edmonds decomposition (D, A, C), and also the same partition of D into vertex sets $D_1, \dots, D_c$ of connected components on D.

Proposition

Let $G_1 \subseteq G_2$ be graphs with the same matching number and on the same vertex set. Then, $D(G_1) \subseteq D(G_2)$ and $D(G_1) \cup A(G_1) \subseteq D(G_2) \cup A(G_2)$.

Step 1: Partitioning into possible Gallai-Edmonds decompositions $(D_1, \ldots, D_c; A; C)$

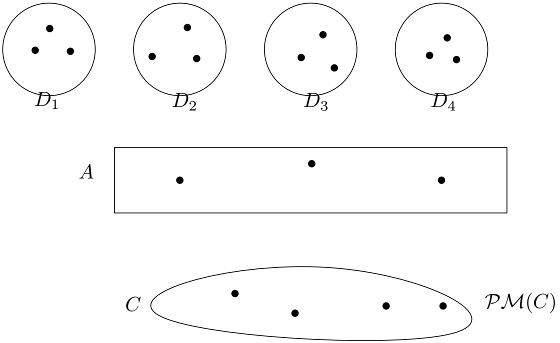
Assume that we have found a Morse matching for each subposet of graphs having the same Gallai-Edmonds decomposition. If there is a directed cycle after combining those Morse matchings, it should look like:



- $D(\sigma_1) = D(\tau_1) \supseteq D(\sigma_2) = D(\tau_2) \supseteq \cdots \supseteq D(\tau_r) \supseteq D(\sigma_1)$.
- $A(\sigma_1) = A(\tau_1) \supseteq A(\sigma_2) = A(\tau_2) \supseteq \cdots \supseteq A(\tau_r) \supseteq A(\sigma_1)$.
- Vertices of each component D_j of D should be the same.

Contradiction!

Step 2: Find a Morse matching for each Gallai-Edmonds decomposition



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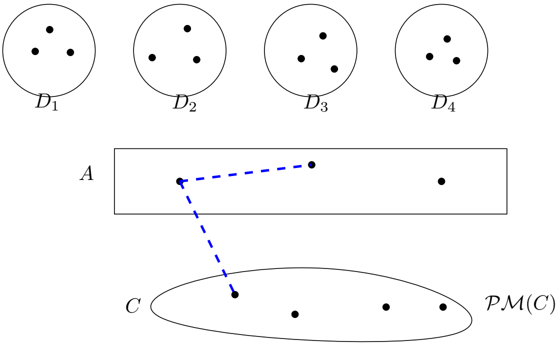
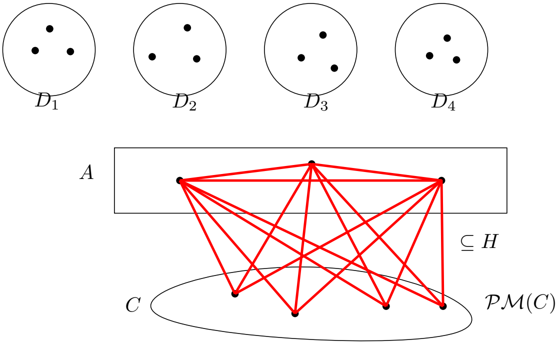


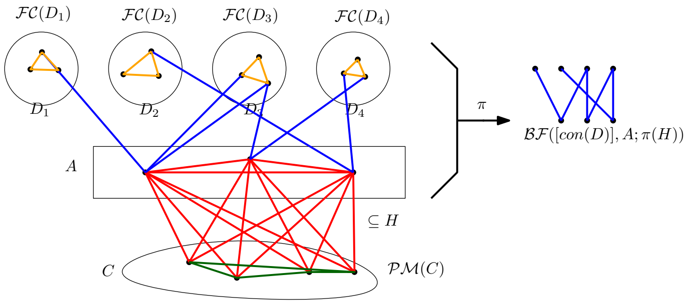
Use pairs $(G+e, G)$ when there is a missing edge e of H between vertices of A, or between a vertex of A and another of C.

Step 2: Find a Morse matching for each Gallai-Edmonds decomposition



We assume that all such edges are in H.

Step 2: Find a Morse matching for each Gallai-Edmonds decomposition

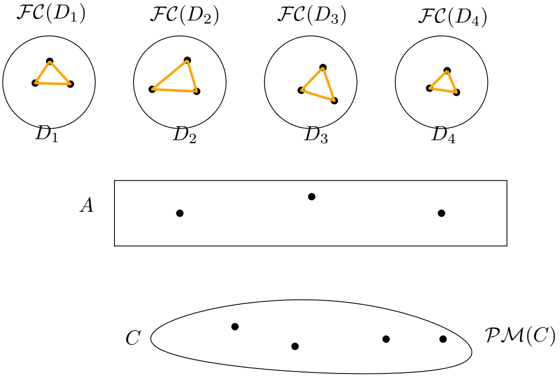


- $\mathcal{P}_1 \times \mathcal{P}_2 := \{\sigma \cup \tau : \sigma \in \mathcal{P}_1,\ \tau \in \mathcal{P}_2\}$.
- $\mathcal{P} = \mathcal{FC}(D_1) \times \mathcal{FC}(D_2) \times \mathcal{FC}(D_3) \times \mathcal{FC}(D_4) \times \mathcal{BF} \times \mathcal{PM}(C)$

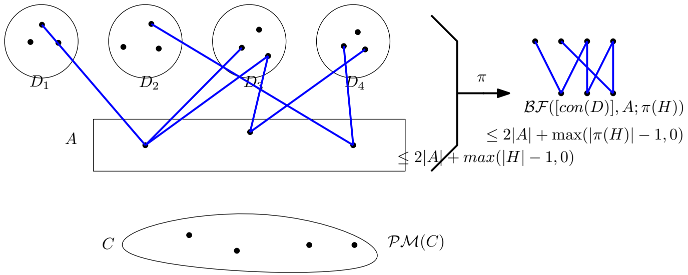
By *Product lemma*, we can (later) combine Morse matchings for each poset in the product, whose critical cells are the union of critical cells for each poset.

Step 2: Find a Morse matching for each Gallai-Edmonds decomposition

$$\leq \sum_{j\in[con(D)]} \tfrac{3}{2}(|D_j|-1) + \max(0, |H[D_j]|-1)$$



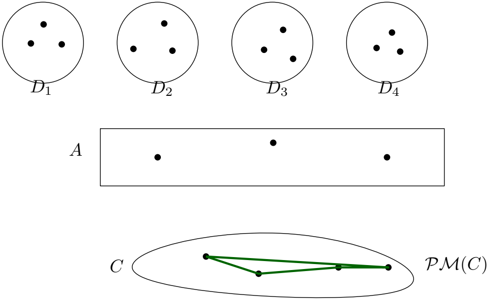
Step 2: Find a Morse matching for each Gallai-Edmonds decomposition



We use *Identification lemma* to obtain a Morse matching $\mathcal{M}$ from the Morse matching $\mathcal{M}_\pi$ of the image. A critical cell G of $\mathcal{M}$ has almost same size with $\pi(G)$ (which is a critical cell of $\mathcal{M}_\pi$) except H-part, that is,

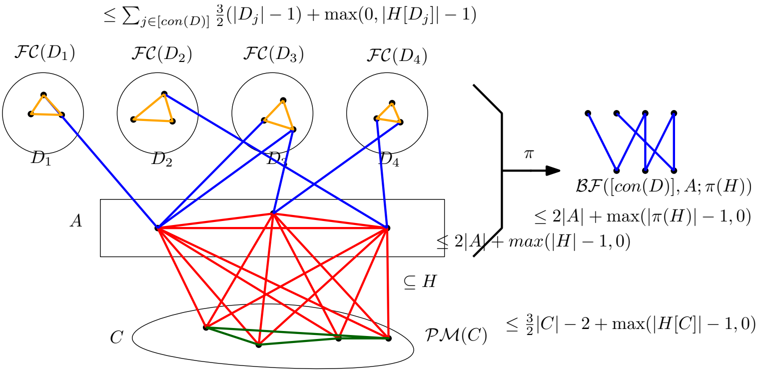
$$|G| = |\pi(G)| - |\pi(H)| + |H|.$$

Step 2: Find a Morse matching for each Gallai-Edmonds decomposition



$\leq \frac{3}{2}|C| - 2 + \max(|H[C]| - 1, 0)$

Step 2: Find a Morse matching for each Gallai-Edmonds decomposition



We use $H[W]$ for $\max(|H[W]|-1, 0)$ when $A \cup C \neq \emptyset$.
When $A \cup C = \emptyset$, there is D_j where $\max(|H[D_j]|-1, 0) = |H[D_j]|-1$.
$\Rightarrow \leq 3k - 4 + |H[[n-1]]|$.

Outline for section 4

Some Questions

- Is there a purely combinatorial proof for the following theorem?

Theorem

Let $E_1, \ldots, E_{3k-2}$ be sets of edges in an arbitrary graph such that

$$\nu(E_i \cup E_j) \geq k \textit{ for every } i \neq j.$$

Then, there exists a rainbow matching of size k.

Some Questions

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Theorem

Let $E_1, \ldots, E_{3k-2}$ be sets of edges in an arbitrary graph such that

$$\nu(E_i \cup E_j) \geq k \text{ for every } i \neq j.$$

Then, there exists a rainbow matching of size k.

- It seems that for many graphs H, $NM_k(H)$ has its non-vanishing homology at only single dimension, but not always. For which H does $NM_k(H)$ have non-vanishing homology in at least two dimensions?

Thank you for your attention!

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any questions?