Rainbows in Fractional Matroid Polytopes

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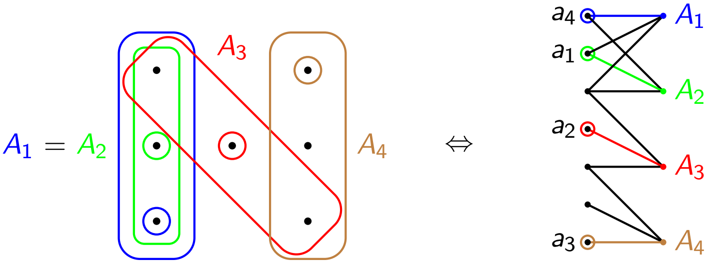
Rainbow Sets

Definition

Given a family $\mathcal{F} = \{A_1, \ldots, A_t\}$ of subsets of V, say $\{a_1, \ldots, a_k\} \subset V$ is *rainbow* if there are distinct $i_1, \ldots, i_k$ such that every $a_{i_j} \in A_j$.

That is, $\{a_1, \ldots, a_k\}$ can be matched in the bipartite incidence graph of $\mathcal{F}$.



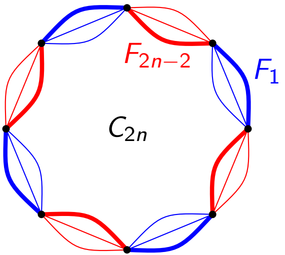
Thanks Xiao for the pictures :-)

Rainbow Matchings

Theorem (Drisko)

Any $2n-1$ n-matchings in a bipartite graph have a rainbow n-matching.

Note that $2n-2$ matchings are insufficient:



Conjecture (Aharoni-Berger)

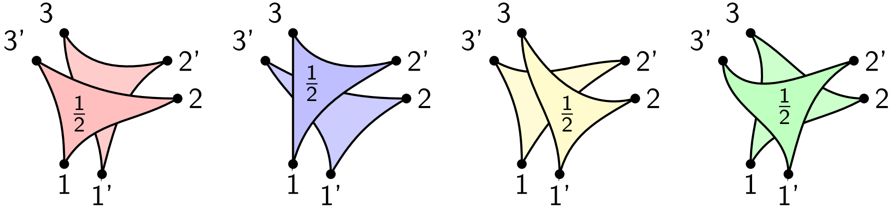
$2n$ n-matchings in *any* graph contain a rainbow n-matching.

Rainbow Fractional Matchings

König: $\nu = \nu^*$ for bipartite graphs.
So, can replace "*n*-matchings" with "*n*-fractional matchings" in Drisko!

Theorem (Aharoni-Holzman-Jiang)

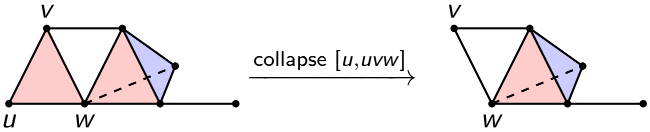
Let H be an r-partite hypergraph, and suppose $F_1, \ldots, F_{rn-r+1} \subset E(H)$ satisfy $\nu^*(F_i) \geq n$. Then $\exists$ rainbow $F \subset E(H)$ satisfying $\nu^*(F) \geq n$.



4 2-matchings in $K^{(3)}_{2,2,2}$ with no integral rainbow 2-matching

d-collapsibility

Let C be a simplicial complex. Say $\sigma \in C^{\leq d}$ is in a unique maximal τ. Let $[\sigma, \tau]$ denote all faces containing σ, and describe its removal from C as an *elementary d-collapse*.



Definition

C is *d-collapsible* if it can be reduced to $\emptyset$ by elementary d-collapses.

Theorem (Kalai-Meshulam)

Say C is d-collapsible. Then any $d+1$ sets $\notin C$ have a rainbow set $\notin C$.

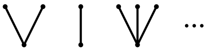
Matroidal Rainbows

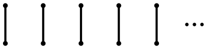
Theorem (Kotlar-Ziv)

Let M_1, M_2 be matroids on V. Then any $2n-1$ n-sets in $M_1 \cap M_2$ have a rainbow set in $M_1 \cap M_2$.

Drisko is the case where both M_1, M_2 are partition matroids:

$F \in M_1 \Leftrightarrow F =$ 

$F \in M_2 \Leftrightarrow F =$ 

$F \in M_1 \cap M_2 \Leftrightarrow F =$ 

Edmonds, AKA "Matroidal König"

Let M, M' be matroids on V. The *fractional matroid polytope* $P(M)$ is given by

$$P(M) := \{f \in \mathbb{R}_+^V : \forall A \subset V, f(A) \leq \mathrm{rk}_M(A)\}.$$

Here $f(A)$ denotes $\sum_A f(x)$.

Lemma

The vertices of $P(M)$ are precisely the indicator vectors $\mathbf{1}_A$ for independent sets $A \in M$.

Theorem (Matroid Intersection Theorem)

The vertices of $P(M) \cap P(M')$ are precisely the indicator vectors $\mathbf{1}_A$ for sets $A \in M \cap M'$ which are independent in both matroids.

Common Generalisation of AHJ and KZ

Write $|f|$ for $f(V)$. This way, $|\mathbf{1}_A| = |A|$.

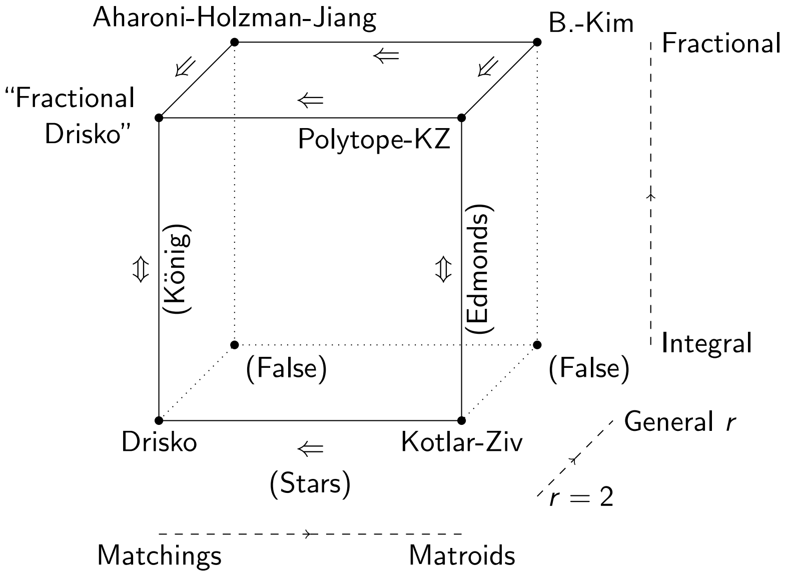
Theorem (B.-Kim)

Let $M_1, \dots, M_r$ be matroids on V. Say $f_1, \dots, f_{rn-r+1} \in \bigcap_i P(M_i)$ all have $|f_j| \geq n$. Then there is an $f \in \bigcap_i P(M_i)$, of size $|f| \geq n$, whose support is rainbow in $\{supp(f_i)\}$.

AHJ showed $C := \{F \subset E(H) : \nu^*(F) < n\}$ is $(rn - r)$-collapsible.
We show the same, redefining ν^* thus:

$$\nu^*(W) := \max\left\{|f| : f \in \bigcap_i P(M_i),\ \mathrm{supp}(f) \subseteq W\right\}.$$

Diagram of Theorems



AHJ-Style Approach

- Choose an *inclusion-minimal* $\sigma := W$ maximising $\nu^*(W) = \bar{n} < n$;
- Perturb LP defining ν^* and $\bigcap P(M_i)$!
  - Unique DP solution $\Leftrightarrow$ W extends to unique maximal τ,
  - Generalise $|f| = \sum f(x)$ to $\sum a_x f(x)$, and reduce:

$$a_x \mapsto \begin{cases} a_x & x \in W \\ a_x - \epsilon & x \in V \backslash W \end{cases}$$

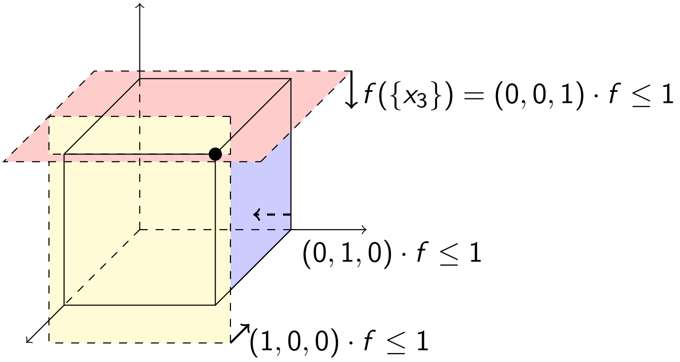
    decreases objective everywhere *except* $[W, \tau]$, allowing induction!
- $|W| \leq rn - r$?!?

Vertices of Polytopes

Let $f \in \bigcap P(M_i)$ be a *vertex* witnessing $\nu^*(W) = \bar{n} = |f|$.

- Minimality of $W \Rightarrow f \neq 0$ on all of W.
- f vertex $\Rightarrow |W|$ L.I. inequalities of $\bigcap P(M_i)$, of form "$f(A) \leq \mathrm{rk}_{M_i}(A)$" are "$=$",

(the only other inequalities defining $\bigcap P(M_i)$ are the "$f(x) \geq 0$"s!)



Say w_i equalities are from M_i, so that $|W| = \sum_i w_i$.

Even More Inequalities at Equality

Since $f(A) = \sum_A f(x)$ and rk is submodular:

$$f(A) + f(B) = f(A \cap B) + f(A \cup B), \text{ and} \tag{1}$$

$$\mathrm{rk}_{M_i}(A) + \mathrm{rk}_{M_i}(B) \geq \mathrm{rk}_{M_i}(A \cap B) + \mathrm{rk}_{M_i}(A \cup B); \text{ hence:} \tag{2}$$

Lemma

$\mathcal{F}_i := \{A \subset V : f(A) = \mathrm{rk}_{M_i}(A)\}$ *is closed under* $\bigcap$ *and* $\bigcup$.

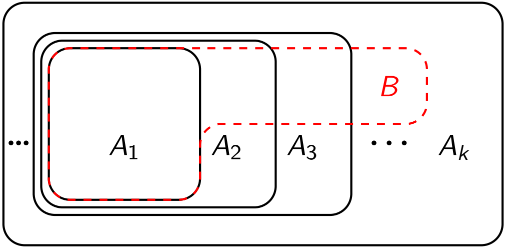
Chain-Building

Lemma

Suppose $\mathcal{F} \subset 2^V$ contains t L.I. sets, and is closed under $\bigcap$ and $\bigcup$. Then $\mathcal{F}$ contains a "chain" $\emptyset \subsetneq A_1 \subsetneq \cdots \subsetneq A_t$ of length t.

Proof: Induct!

While $k < t$, choose $B \in \mathcal{F} \setminus \text{span}\{A_i\}_{i=1}^{k}$.



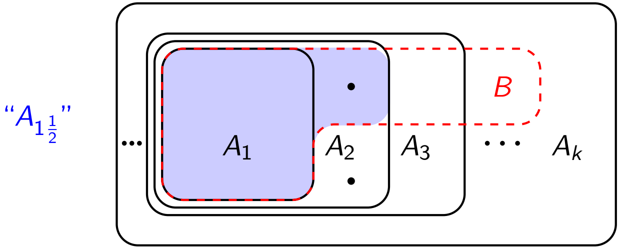
Proof of Chain-Building

Given $\emptyset =: A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k$, for some $k < t$, let $B \in \mathcal{F} \backslash \mathrm{span}\{A_i\}_{i=1}^k$.

Is there a j, with both $(A_{j+1} \setminus A_j) \cap B \neq \emptyset$ AND $(A_{j+1} \setminus A_j) \setminus B \neq \emptyset$?

If YES, insert:

$$\ldots A_{j-1} \subsetneq A_j \subsetneq \underbrace{A_j \cup (A_{j+1} \cap B)}_{\in \mathcal{F}} \subsetneq A_{j+1} \subsetneq A_{j+2} \ldots$$



If NO, $B = \bigcup_S (A_{j+1} \setminus A_j) \Rightarrow \mathbf{1}_B = \sum_S (\mathbf{1}_{A_{j+1}} - \mathbf{1}_{A_j})$, contradicting $B \notin \mathrm{span}\{A_1, \ldots, A_t\}$!

$|W| \leq rn - r$

In $\mathcal{F}_i := \{A : f(A) = \mathrm{rk}_{M_i}(A)\}$, there is a chain

$$\emptyset \subsetneq A_1 \subsetneq \cdots \subsetneq A_{w_i}.$$

But, as $f|_W > 0$:

$$\begin{array}{ccccc} 0 < & f(A_1) & < \cdots < & f(A_{w_i}) & \leq f(V) = |f| < n \\ & \| & & \| & \\ & \mathrm{rk}_{M_i}(A_1) & \cdots & \mathrm{rk}_{M_i}(A_{w_i}) & \\ & & & \mid\!\vee & \\ & & & w_i, & \end{array}$$

so $w_i \leq n - 1$ as it is an integer. Hence $|W| = \sum_{i \in [r]} w_i \leq rn - r$.

Perturbation Restrictions

Given $b : 2^V \to \mathbb{R}_+$, and matroid M on V, the skew matroid polytope is

$$P_b(M) := \{f \in \mathbb{R}_+^V : \forall A \subset V, f(A) \leq b(A)\mathrm{rk}(A)\}.$$

Alas, the submodularity condition (2) fails for general $b(A) \cdot \mathrm{rk}(A)$! One can't perturb b arbitrarily. Must balance:

- Keeping $\{b\}$'s general enough to allow perturbations, and
- Restricting to only b's for which $b \cdot \mathrm{rk}$ is submodular.

Already desire b to be decreasing, for

$$f(A_1) = b(A_1)\mathrm{rk}(A_1) < \cdots < b(A_t)\mathrm{rk}(A_t) = f(A_t)$$

to imply

$$\mathrm{rk}(A_1) < \cdots < \mathrm{rk}(A_t).$$

An Old Exercise from Undergraduate Analysis

Lemma (Folklore)

Suppose f, g are convex functions, with f increasing and g decreasing. Then $f \cdot g$ is convex.

Lemma (...stackexchange?)

Suppose f, g are submodular functions, with f increasing and g decreasing. Then $f \cdot g$ is submodular.

Proof: For $\Delta_Y(f)(X) := f(X \cup Y) - f(X)$, write

$$f \cdot g(A \cup B) + f \cdot g(A \cap B) - f \cdot g(A) - f \cdot g(B) = \Delta_{B \setminus A} \Delta_{A \setminus B}(f \cdot g)(A \cap B),$$

and apply a "product rule for discrete derivatives".

Suggests restricting such $b : 2^V \to \mathbb{R}_+$ to functions which are Positive, Decreasing, and Submodular (or "PDS").

PDS Functions are Still Sufficiently General

Thus, to allow the perturbations, it suffices to prove:

Lemma

The cone

$$Q := \left\{ b \in \mathbb{R}^{2^V} : b \ PDS \right\}$$

has full dimension.

To see this, let $b(A) := 2|V|^2 - |A|^2$ for every $A \in 2^V$. Then b is strictly positive, strictly decreasing, and strictly submodular! So it's in the interior of Q.

Thank you