Math++ Problems

Problem set 3 – Representations of finite groups

hints after **13. 5. 2020**, solutions due **20. 5. 2020**

Definition: S_n denotes the symmetric group of permutations of $\{1, \ldots, n\}$.

Definition: Unless stated otherwise, we will use the inner product given by $\langle \varphi, \psi \rangle = \mathbb{E}_{g \in G}\, \varphi(g)\overline{\psi(g)}$ for $\varphi, \psi\colon G \to \mathbb{C}$.

1. Find a representation $g\colon G \to \mathrm{GL}(V)$ with an invariant subspace $W \subset V$ such that $W^{\perp}$ is not invariant. **[1]**

2. Let $\rho\colon G \to \mathrm{GL}(\mathbb{C}^n)$ be an irreducible representation. Let R_g be a matrix of $\rho(g)$ with respect to a fixed basis. Assume that all R_g are unitary for every $g \in G$. Let $r_{i,j}(g)$ be the entry of R_g at position i, j. Note that $r_{i,j}$ is a function $G \to \mathbb{C}$ for i and j fixed. Prove that $\langle r_{i,j}, r_{k,l} \rangle = 1/n$ for $(i, j) = (k, l)$ and $\langle r_{i,j}, r_{k,l} \rangle = 0$ otherwise. **[1]**

3. (Maschke's theorem over a field of characteristic 0.) Let G be a finite group, $\mathbb{T}$ be a field of characteristic 0 and $\rho\colon G \to \mathrm{GL}(V)$ be a representation with $\dim V < \infty$. Prove that if U is an invariant subspace of V, then there is an invariant subspace W such that $V = U \oplus W$. **[2]**
   Hint: Define W as the kernel of a suitably weighted projection $V \to U$.

4. Let $\rho\colon G \to \mathrm{GL}(\mathbb{C}^n)$ be an irreducible representation of a finite group G with character χ. Let C be the centre of G (that is $C := \{s \in G \mid (\forall g \in G)(sg = gs)\}$).
   (a) Prove that for every $s \in C$ there is $c_s \in \mathbb{C}$ such that $\rho_s = c_s I_n$. Deduce that $|\chi(s)| = n$ for every $s \in C$. **[1]**
   (b) Prove that $n^2 \leq |G| \,/\, |C|$. **[1]**
   (c) Assume that $\rho_s \neq I_n$ for every $s \neq 1_G$. Show that C is a cyclic group. **[2]**

5. Find all irreducible representations of the cyclic group of order $n \in \mathbb{N}$ and their characters. Check that this way, you get exactly the characters which you know from commutative Fourier analysis. Check also that the formula for non-commutative Fourier analysis agrees with the formula for commutative one in this case. **[1]**

6. Let $\rho\colon S_n \to \mathrm{GL}(\mathbb{C}^n)$ be the permutation representation of S_n. Let $W := \{v \in \mathbb{C}^n \mid \sum_i^n v_i = 0\}$. Prove that the restriction of ρ to W is an irreducible representation of S_n for every $n \geq 2$. **[2]**

7. Prove that $\widehat{\varphi * \psi}(\rho) = \widehat{\varphi}(\rho)\widehat{\psi}(\rho)$ for a representation $\rho\colon G \to \mathrm{GL}(\mathbb{C}^n)$. **[1]**

8. Given $\pi \in S_n$ let us define $\lambda(\pi) = (\lambda_1, \ldots, \lambda_k)$ as the vector of lengths of all cycles in π order in a nonincreasing sequence.
   (a) Prove that two permutations $\pi, \sigma \in S_n$ are conjugate if and only if $\lambda(\pi) = \lambda(\sigma)$. **[1]**
   (b) Describe the Specht module S^{λ} and the action of S_n on it explicitly for $\lambda = (n)$, $\lambda' = (1, \ldots, 1)$ and $\lambda'' = (n-1, 1)$. **[2]**