Mathematics++ (topology), lecture #2 (March 15, 2021)

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Separation axioms

Last time: basic definitions, some examples of topological spaces.

Remark connectednes... invariant under taking homeomorphisms.

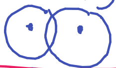
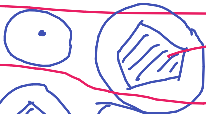
Example S^1, $(0,1)$, $[0,1]$ are not homeomorphic

removing one point, we get a connected space from S_1

from $(0,1)$ ... disconn.

$[0,1]$ ... sometimes connected, sometimes disc.

Separation axioms

- T_0 ($\forall x \neq y$ $\exists U$ open s.t. $x \in U$ and $y \notin U$ or vice versa)
- T_1  $\forall x \neq y$: each pt. has a neighborhood not containg the other pt
- T_2  $\forall x \neq y$ $\exists$ disjoint nbhds Hausdorff Space
- T_3 T_2 &  closed sets (regular)
- T_4 T_2 &  (normal)

Remark There others, terminology not unified

listed in increasing strength $T_{i+1} \Rightarrow T_i$ $\forall i$

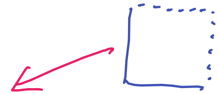
but this hierarchy strict: $\forall i$ T_i, T_{i+1} not equivalent

Separability

Definition $D \subseteq X$ is *dense* if $\mathrm{cl}\, D = X$.

X is *separable* if it has countable dense subset.

Example
- $\mathbb{Q}^2 \subseteq \mathbb{R}^2$ is countable dense subset
  $\forall$ subspace of $\mathbb{R}^2$ also separable
- Sorgenfrey plane ... is separable, has non-separable subspaces



Theorem (Tieze extension theorem) X a T_4, $A \subseteq X$ closed $f: A \to \mathbb{R}$ continuous. Then $\exists$ *continuous extension* $\bar{f}: X \to \mathbb{R}$

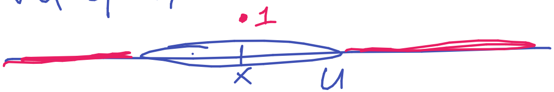
s.t. $\sup_{x \in X} |\bar{f}(x)| \leq \sup_{x \in A} |f(x)|$.

Theorem (Urysohn metrization theorem) X is T_3, X has countable base $\Rightarrow$ X is metrizable.

X T_3 with countable base $\Rightarrow$ X is T_4

Lemma X is T_4 space w/ countable base $\Rightarrow$ $\exists$ a countable sequence $(f_1, f_2, \ldots)$ of ct. functions $X \to [0,1]$ s.t.

$\forall x \in X$ $\forall U$ open, $x \in U$ $\Rightarrow$ $\exists f_i$ s.t. f_i is $\begin{cases} 0 & \text{outside } U \\ 1 & \text{in } x. \end{cases}$



Lemma

X is a T_4 space with countable base $\Rightarrow \exists$ a countable sequence $f_1, f_2, \dots$ of continuous functions from X to $[0,1]$ such that: whenever $x \in X$, and U open set with $x \in U$, there is f_i such that f_i is 0 outside U and 1 in x.

Proof. Easy to show: $\{x\}$ is closed, $\{x\} \cup (X \setminus U)$ closed

Define $f = \begin{cases} 1 & \text{in } x \\ 0 & \text{in } X \setminus U \end{cases}$ use extension thm.

But this is not countable sequence, have to be smarter

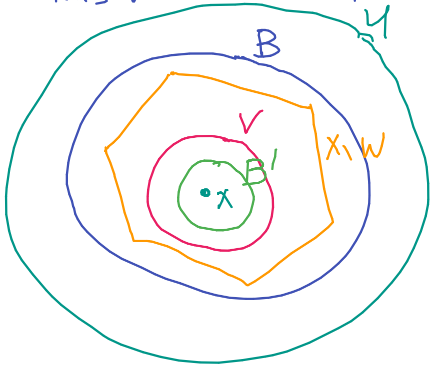
For every pair of sets B', B $\quad cl B' \subseteq B$ take f on $\boxed{cl B' \cup (X \setminus B)}$ $f(y) = \begin{cases} 1 & cl B' \\ 0 & \text{otherwise} \end{cases}$

This works: Fix x, U. Find B s.t. $x \in B \subseteq U$. X is $T_3 \Rightarrow$ find V, W open, disj. s.t. $x \in V$, $X \setminus B \subseteq W$.

Also find B' in basis, $x \in B' \subseteq V$. $B' \subseteq X \setminus W$

$X \setminus W$ is closed $\Rightarrow$ $cl B' \subseteq X \setminus W \subseteq B \subseteq U$.

$\Rightarrow$ for given x, U, we found the appropriate function.



These f are continuous:
f is defined on union of closed sets:

Proof of Urysohn's metrization theorem

Theorem

X is T_3 (T_4), has countable base $\Rightarrow$ X is metrizable.

Proof. We will show that X is homeomorphic to a subspace of H... Hilbert cube

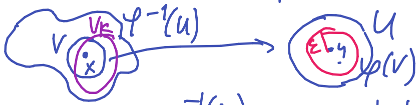
$H :=$ all infinite sequences $(x_1, x_2, x_3, \dots)$, where $\forall i$ $\quad x_i \in [0, \frac{1}{i}]$

with ℓ_2 metric, i.e. $dist(x,y) = \sqrt{\sum (x_i - y_i)^2}$

Define $\varphi : X \longmapsto \left(\frac{1}{1} f_1(x), \frac{1}{2} f_2(x), \frac{1}{3} f_3(x), \dots\right)$

Need: f continuous
f injective
f^{-1} continuous

φ is cont.:



Let $U \subseteq H$ open, want $\varphi^{-1}(U)$ open. Let $x \in \varphi^{-1}(U)$, let $y := \varphi(x)$. want: $\exists V$ open, $x \in V$, $\varphi(V) \subseteq U$. Let ε be s.t. $B(y,\varepsilon) \subseteq U$. Suffices to find V s.t. $\varphi(V) \subseteq B(y,\varepsilon)$.

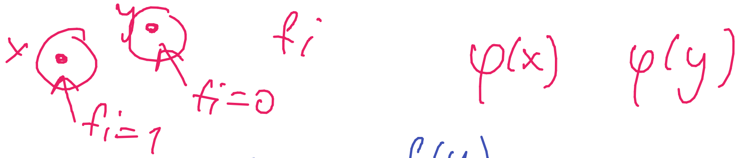
Find n_0 s.t. $\sum_{k \geq n_0} \frac{1}{k^2} < \frac{\varepsilon^2}{\cancel{4}\ 8}$

For $k < n_0$, let $V_k := \left\{ z \in X ;\ \left(\frac{1}{k} f_k(z) - y_k\right)^2 < \frac{\varepsilon^2}{2 \cdot n_0} \right\}$ is open since f_k cont.

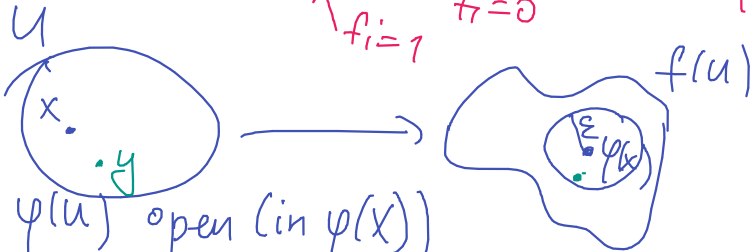
Then $V := \bigcap_{k<n_0} V_k$ is also open, and for $z \in V$, we have

$$\|\varphi(z) - y\|^2 = \underbrace{\sum_{k<n_0} \underbrace{\left(\frac{1}{k} f_k(z) - y_k\right)^2}_{\leq \frac{\varepsilon^2}{2 \cdot n_0}}}_{\leq \frac{\varepsilon^2}{2}} + \underbrace{\sum_{k \geq n_0} \underbrace{\left(\frac{1}{k} f_k(z) - y_k\right)^2}_{\leq \left(\frac{2}{k}\right)^2}}_{\leq \frac{\varepsilon^2}{2}} \leq \varepsilon^2$$

φ injective ...... clear



φ^{-1} is continuous

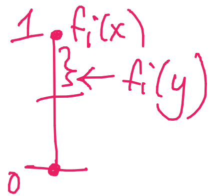


$U \subseteq X$ open. Want $\varphi(U)$ open (in $\varphi(X)$)

That is, $\forall x \in U \ \exists \varepsilon \quad B(\varphi(x), \varepsilon) \subseteq \varphi(U)$.

Find i with $f_i(x) = 1$, $f_i = 0$ outside U. Let $\varepsilon = \frac{1}{2i}$

If y is s.t. $\|\varphi(x) - \varphi(y)\| < \varepsilon \Rightarrow \left| \frac{1}{i} \underbrace{f_i(x)}_{1} - \frac{1}{i} f_i(y) \right| < \varepsilon = \frac{1}{2i}$



$\Rightarrow \frac{1}{i} f_i(y) > \frac{1}{2i} > 0$

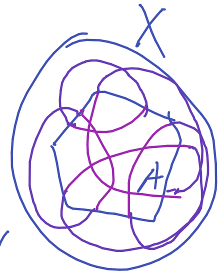
$\Rightarrow y$ is not outside U. $\Rightarrow y \in U \Rightarrow \varphi(y) \in \varphi(U)$ ∎

Compactness

Definition X is compact if whenever $\mathcal{U}$ is a collection of open sets whose union is all of X, $\exists$ finite $\mathcal{U}_0 \subseteq \mathcal{U}$ whose union is X.

"every open cover has finite subcover"

$C \subseteq X$ is compact if C with subspace topo is compact.



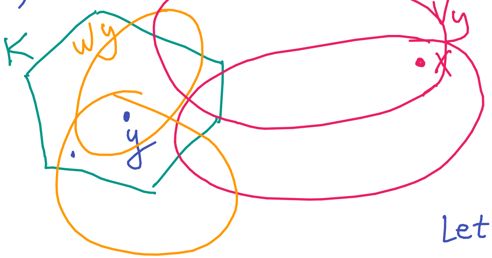
Lemma (i) X compact, $A \subseteq X$ closed $\Rightarrow$ A compact

(ii) X Hausdorff, $K \subseteq X$ compact $\Rightarrow$ K closed

(iii) $f : X \to Y$ ct., $K \subseteq X$ compact $\Rightarrow$ $f(K)$ compact.

Proof. (i) Let $\mathcal{U}$ open cover of A. For every $U \in \mathcal{U}$ $\exists$ $\tilde{U}$ open in X, $U = \tilde{U} \cap A$. $\{\tilde{U} ; U \in \mathcal{U}\} \cup \{X \setminus A\}$ is an open cover of X $\Rightarrow$ $\exists$ finite subcover. Restrictions of these sets to A form open subcover of $\mathcal{U}$.

(ii) Need to show: $X \setminus K$ open. Take $x \in X \setminus K$. Want an open set around x, $\subseteq X \setminus K$.



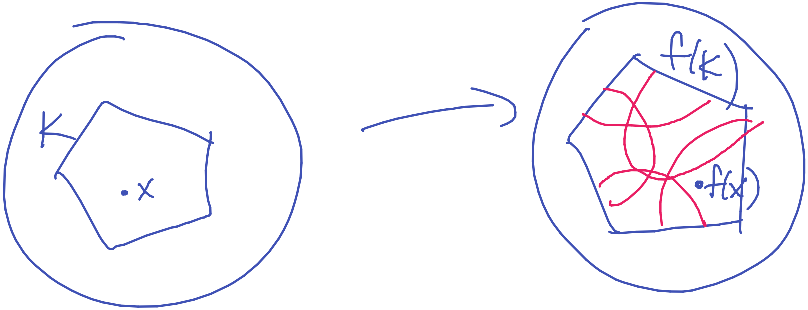
X is T_2 $\Rightarrow$ $\forall y \in K$ $\exists$ open sets V_y and W_y disjoint, $y \in W_y$, $x \in V_y$.

Sets W_y ... a cover of K.

Take finite subcover $W_{y_1}, \dots, W_{y_n}$.

Let $V := \bigcap_{i=1}^{n} V_{y_i}$. This is open, disj. from K.

(iii).



Take $\mathcal{U}$ open cover of $f(K)$. If $x \in K$ $\exists U \in \mathcal{U}$ s.t. $f(x) \in U$.

$\Rightarrow \{f^{-1}(U) ; U \in \mathcal{U}\}$ is open cover of K.

Let $f^{-1}(U_1), \dots, f^{-1}(U_n)$ finite subcover.

$\Rightarrow U_1 \dots U_n$ finite subcover of $\mathcal{U}$. $\square$

Compactness II

Theorem (Continued real-valued function on compact set attains its minimum)

Proof.

$\square$

Products of spaces

Definition

Remark

Example

Exercise

Theorems of Tychonoff and de Bruijn—Erdős (about graph colorings)

Theorem (Tychonoff)

Theorem (de Bruijn—Erdős)

Proof.