Non-embeddability

which graphs are planar?

G is understood as simpl. complex

$f : |G| \to \mathbb{R}^2$; embedding (a homeomorphism to $f(|G|)$)

There are complete answers: G is planar

$\Updownarrow$

G has no subdivision of $K_{3,3}$ / K_5

$\Updownarrow$

G has no $K_{3,3}$-minor & no K_5-minor.

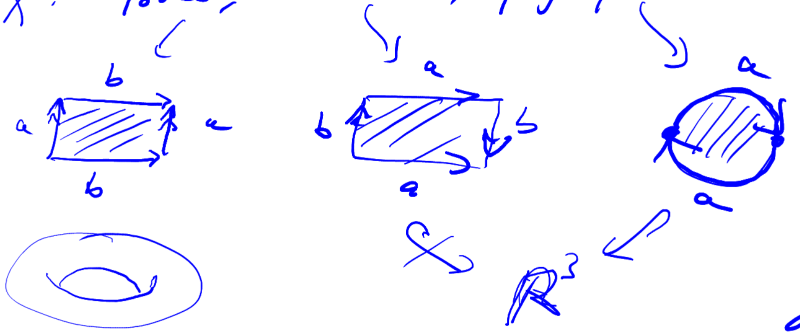
no!

In general X, Y topol. spaces

Q: $\exists f : X \to Y$ embedding

X compact Y Hausdorff $\Rightarrow$ we just need $f : X \to Y$ injective continuous

X: torus, Klein bottle, proj. plane



$X \hookrightarrow \mathbb{R}^3$

1-1 mapping

no complete answers!

given X — finite 4-dim. simpl. complex

$Y = \mathbb{R}^5$

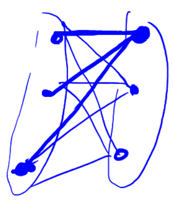
it is undecidable whether $X \hookrightarrow Y$

Recall: k-dimens. finite simpl. complexes always embeds into $\mathbb{R}^{2k+1}$

Thm (Van Kampen & Flores) $\forall k \geq 1$ $\exists$ k-dim. fin. simpl. comp. $C_k \not\hookrightarrow \mathbb{R}^{2k}$

$D_3 :=$ ⋮ 3 isol. vertices

$$C_k = \underbrace{D_3 * D_3 * \dots * D_3}_{k+1}$$



D_3 D_3

$C_2 \cong K_{3,3}$

imagine cont. inj. $f : K \to \mathbb{R}^d$, want contradiction

$\forall x \neq y$ $f(x) \neq f(y)$

From $|K|$ we construct new spaces & mappings that will be antipodal $\longrightarrow$ Borsuk-Ulam theorem

Abstract version of antipodality

concrete $x \mapsto -x$

objects: $\mathbb{Z}_2$-space: (X, ν), X top. sp., $\nu : X \to X$, $\nu \circ \nu = id_X$

morphisms: $\mathbb{Z}_2$-map $(X,\nu) \to (Y, \omega)$

$f : X \xrightarrow{f} Y$ cont.

s.t.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \nu \downarrow & \circ & \downarrow \omega \\ X & \xrightarrow{f} & Y \end{array}$$

$f \circ \nu = \omega \circ f$

$\nu : X \to X$, $\omega : Y \to Y$ are homeomorphisms

(if $\nu(x) = -x$: $f(-x) = -f(x)$)

Deleted product

$K \rightsquigarrow |K|^2_\Delta$

$X \rightsquigarrow X^2_\Delta := \{(x,y) : x, y \in X,\ x \neq y\}$

X^2_Δ has $\mathbb{Z}_2$-map $(x,y) \mapsto (y,x)$

$f : K \to \mathbb{R}^d \rightsquigarrow \tilde{f} : |K|^2_\Delta \to S^{d-1}$

$$\tilde{f}(x,y) = \frac{f(x) - f(y)}{\underbrace{\| f(x) - f(y) \|}_{\neq 0}}$$

& $\mathbb{Z}_2$-map

well-defined

$\| \tilde{f}(x,y) \| = 1$

$\exists$ embedding

we proceed: if $|K| \hookrightarrow \mathbb{R}^d$ then $|K|^2_\Delta \xrightarrow[\mathbb{Z}_2\text{-mapps}]{} S^{d-1}$

$\boxed{\text{Prop.}}$ K fin. simpl. complex s.t. there is no $\mathbb{Z}_2$-map $|K|^2_\Delta \to S^{d-1}$.

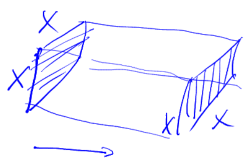
Then $|K| \not\hookrightarrow \mathbb{R}^d$.

Deleted joins

.... aim: remove some points from $X^{*2} = X * X$

so to make an analogy of Gaussian map well def. on $X^{*2} \setminus$?

$X^{*2} = X \times X \times [0,1]/\sim$



$f : X \hookrightarrow \mathbb{R}^d$ inj.

$$\tilde{f}(x,y,t) := \frac{(t f(x) - (1-t) f(y),\ 2t-1)}{\| \text{---} \|} \in \mathbb{R}^{d+1}$$

$\text{---} = 0 \Rightarrow t = \frac{1}{2}$

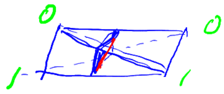
& $\frac{1}{2} f(x) - \frac{1}{2} f(y) = 0$

$\Rightarrow f(x) = f(y)$

$\Rightarrow x = y$

$\tilde{f}(x,y,t)$ is well-def'd, unless $t=\frac{1}{2}$ & $x=y$ $X =$  $X^{*2} \simeq$ simplex

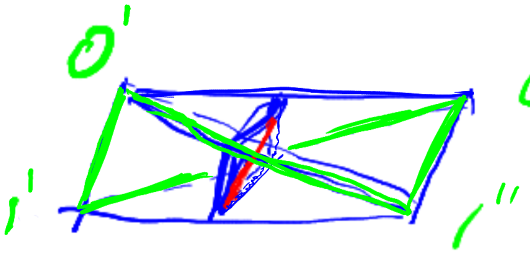
If it is well-def'd, then $\|\tilde{f}(x,y,t)\| = 1.$



Thus, we define ① $\underline{X^{*2}_{\Delta}} = X^{*2} \setminus \{(x,x,\tfrac{1}{2}) ; x \in X\}$ X ... top. space

$\longrightarrow$ a $\mathbb{Z}_2$-space $(x,y,t) \mapsto (y,x,1-t)$

② <u>simplicial deleted join</u> K ... simpl. complex $(|K| = X)$



$\underline{K^{*2}_{\Delta}} := \{F' \cup G'' : F, G \in K, \ \underline{F \cap G = \emptyset}\}$

two copies of K: K' & K''

$K = \{\{0\}, \{1\}, \{0,1\}, \emptyset\}$

$\longrightarrow |K^{*2}_{\Delta}|$ is a $\mathbb{Z}_2$-space $0' \leftrightarrow 0''$, $1' \leftrightarrow 1''$

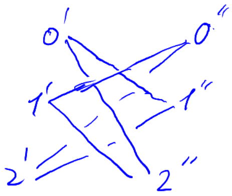
$\longrightarrow \underset{②}{|K^{*2}_{\Delta}|} \subseteq \underset{①}{|K|^{*2}_{\Delta}}$

$\boxed{\text{Prop.}}$ K fin. simpl. compl.

IF $\nexists$ no $\mathbb{Z}_2$-map $|K^{*2}_{\Delta}| \to S^d \ (\subset \mathbb{R}^{d+1})$. THEN $|K| \not\hookrightarrow \mathbb{R}^d$

$\boxed{\text{For } K = D_3^{*(k+1)}, \text{ what is } \ ?}$

1) $k=0 \Rightarrow K = D_3$



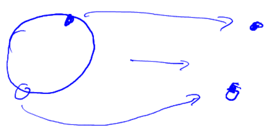
$|K^{*2}_{\Delta}|$ is a (6-)cycle $\simeq S^1$

2) $(M * L)^{*2}_{\Delta} \simeq M^{*2}_{\Delta} * L^{*2}_{\Delta}$

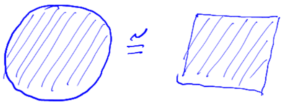
$\underline{K^{*2}_{\Delta}} \simeq \underbrace{(D_3)^{*2}_{\Delta} * \underbrace{(D_3)^{*2}_{\Delta} * \cdots\cdots}}_{k+1} \simeq \underbrace{S^1 * \cdots * S^1}_{k+1}$

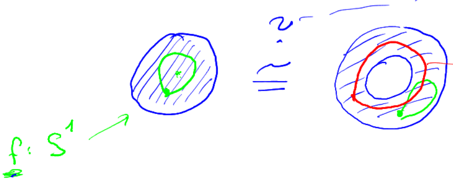
$$|K_\Delta^{*2}| \simeq \underline{S^{2k+1}} \xrightarrow[\text{no antip. mapping}]{\not\quad} S^{2k} \Longrightarrow |K| \not\hookrightarrow \mathbb{R}^{2k}$$

by Borsuk-Ulam $\blacksquare$



Homotopy groups

Question :  — we prefer $\not\cong$



h is NOT null-homotopic

null-homotopic map $f: X \to Y$: $f \sim g$ & g is a constant mapping

homotopy

$H(x,t) = t \cdot f(x)$

$H(x,1) = f(x)$

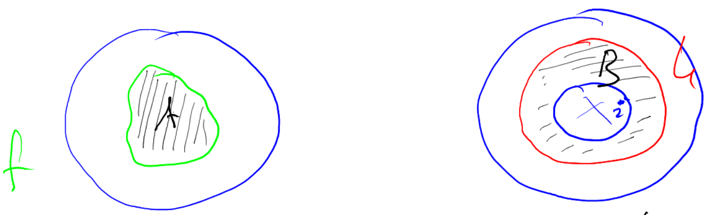
$H(x,0) = 0$ $\longrightarrow$ HOMOTOPY GROUPS

ANOTHER DIFFERENCE of f & h $\longrightarrow$ HOMOLOGY GROUPS

(of )

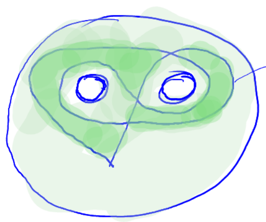
f is a boundary of something

h is not



f is boundary of A

boundary of B is $h + i$



f is not null-homotopic

but f is a boundary

Pointed spaces et al.

loop with basepoint $x_0 \in X$

$a: [0,1] \to X$

$a(0) = a(1) = x_0$

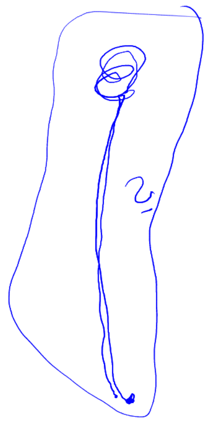
$b: [0,1] \to X$

$c: [0,1] \to X$

a & b $\rightsquigarrow$ $c = a \cdot b$

$$c(t) = \begin{cases} a(2t) & \text{if } t \in [0, \tfrac{1}{2}] \\ b(2t-1) & \text{if } t \in [\tfrac{1}{2}, 1] \end{cases}$$

$c: [0,1] \to X$

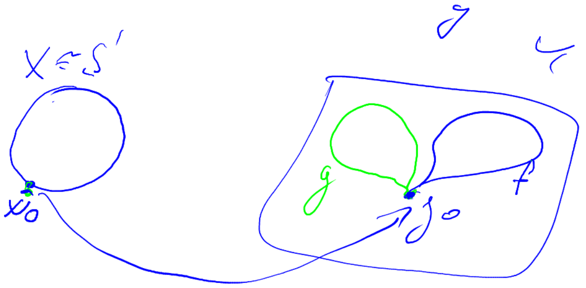


pointed space : (X, x_0)

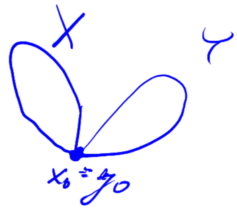
top. sp. $x_0 \in X$

pointed map $(X, x_0) \xrightarrow{f} (Y, y_0)$ s.t. $f: X \to Y$ is cont.
& $f(x_0) = y_0$

pointed homotopy – $(X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$ $f \sim g$ iff $\exists H : X \times [0,1] \to Y$
s.t. $H(x,0) = f(x)$
$H(x,1) = g(x)$
& $H(x_0, t) = y_0$



pointed products , pointed disj. union = wedge
$(X, x_0) \vee (Y, y_0)$



Remark   pair (X, A) top. sp. $A \subseteq X$
(subcomplex of fin. simpl. complex)

$f : (X, A) \longrightarrow (Y, B)$
$f : X \to Y$ is cont.
& $f(A) \subseteq B$

$\longrightarrow$ relative homotopy / homology

$\boxed{\text{The fundamental group} \quad \pi_1(X)}$ X ... pointed space
(Poincaré)

Df (X, x_0) ... pointed space

$\pi_1(X, x_0)$ is a group with elements $[(S^1, s_0), (X, x_0)]_*$

pointed homotopy classes   pointed space

& operation by composition : $[a]_* \cdot [b]_* = [a * b]_*$

(unit element 1 : constant map to x_0)

1) Well-defined ——— $a_1 \sim a_2$ & $b_1 \sim b_2 \Rightarrow a_1 \cdot b_1 \sim a_2 \cdot b_2$

2) 1 is a unit element

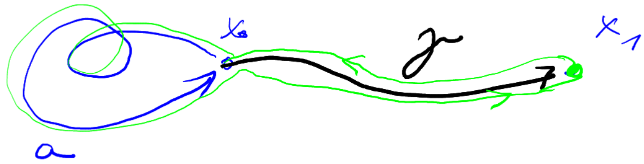
3) $\cdot$ is associative

Exercise X top-space, x_0, x_0' — two points connected by a path $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$

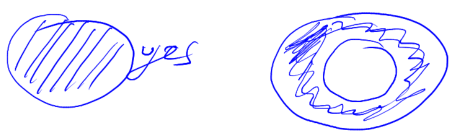
Then

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1)$$

$$a \longrightarrow b$$



$\to$ simply connected space: X s.t. $\pi(X)$ is trivial (one element)



not s. conn.

$\boxed{\text{Functor}}$ $f : (X, x_0) \to (Y, y_0)$ pointed map

$\Downarrow$

$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ group homomorphism

$f_*([a]_*) = [fa]_*$ ——→ well-defined & group homom.

$$\begin{tikzcd}
(S^1, s_0) \arrow[r, "a"] \arrow[rr, bend right, "fa"'] & (X, x_0) \arrow[r, "f"] & (Y, y_0)
\end{tikzcd}$$

& $(fg)_* = f_* \cdot g_*$

π_1 is a functor from category of pointed spaces to — " — of groups.

$\boxed{\text{Df}}$ Functor F from $\mathcal{C}$ to $\mathcal{D}$ maps $X \in Ob(\mathcal{C})$ to $F(X) \in Ob(\mathcal{D})$ and $f \in Hom(X, Y) \longrightarrow F(f) \in Hom(F(X), F(Y))$

s.t. $F(id_X) = id_{F(X)}$

& $F(fg) = F(f)\,F(g)$

another functor that we need: $K \longrightarrow |K|$

[simplicial complexes $\longrightarrow$ top. spaces]