simplicial complexes — abstract -- set system

— geometric --- collection of simplices w same props.

simplicial mappings between abstract simplicial complexes $s: V(K)\to V(L)$, $\forall \sigma\in K$ $s(\sigma)\in L$

⤳ generate a canonical cont. map between the geometric realizations $|s|:|K|\to|L|$

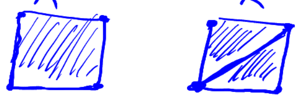
- $|s|$ is continuous
- s is injective $\Rightarrow$ $|s|$ is injective
- s is isomorphism $\to$ $|s|$ is homeomorphism

$id_K: K\to K \Rightarrow |id_K|$ is homeo $|K|\to|K|$ — "identity"

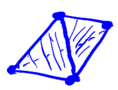
- $|K|$ is uniquely def'd, up to homeomorphism
- space X has triangulation K $\Leftrightarrow$ $X\cong|K|$

(?) $\boxed{\exists \text{ spaces with no triangulation}}$ ... 4-dim. manifold

X K



$[0,1]^2$



• triangulation of S^{n-1} in $\mathbb{R}^n$: boundary of n-dim. simplex

$\sigma = conv\{v_0,\dots,v_n\}$

all faces of σ, except σ itself

• $\exists$ triangulation of a torus with 7 vertices 14 triangles & this is best possible

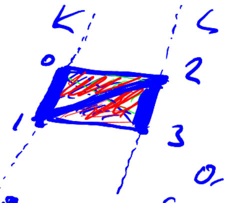
simplicial join of K, L: 1) assume $V(K)\cap V(L)=\emptyset$

$V(K*L)=V(K)\cup V(L)$

$K*L=\{F\cup G : F\in K, G\in L\}$

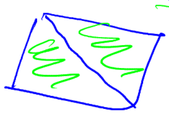
2) if $V(K)\cap V(L)\neq\emptyset$, then first replace L with some L'

0, 1, 01 K L 2, 3, 23



0, 1, 01, ...

$K*L=\{012, 013, 023, 123,$



$0123\}$

"!" $|K*L|\cong|K|*|L|$ → $v_0\dots v_k$ → $w_0\dots w_\ell$

if K is k-simplex, L is ℓ-simplex $\Rightarrow$ $K*L$ is $(k+\ell+1)$-simplex → $\{v_0\dots v_k, w_0\dots w_\ell\}$ & all subsets

$|K|, |L|$ ... geometric simpl.

$|K|*|L|$ is geom. simplex

$\to S^n \cong \underbrace{S^0 * S^0 * \cdots * S^0}_{n+1}$ (sphere; a_0, b_0; a_1, b_1; … ; a_n, b_n)

$V(S^0) = \{a, b\}$ $\quad S^0 : \{\{a\}, \{b\}, \emptyset\}$

one poss. triangulation of S^n

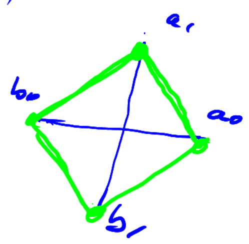
$V(S^n) = \{a_0, \dots, a_n, b_0, \dots, b_n\}$

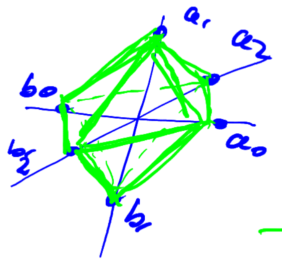
$S^n = \{F \subseteq V(S^n) : \text{for no } i : \{a_i, b_i\} \subseteq F\}$

$a_i \to e_i = (0010\dots0)$

$b_i \to -e_i$

$n=1$





octahedron

$\longrightarrow$ cross polytope

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combinatorial examples

1) clique-complex / flag-complex of a graph

a simple undir. graph $G \rightsquigarrow C(G)$: vertices : $V(G)$

simplices : complete subgraphs

independence complex $\rightsquigarrow$ Aharoni-Haxell $\rightsquigarrow$ Ryser conjecture

2) order complex

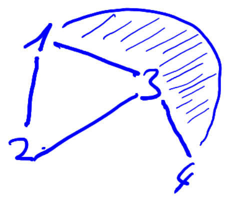
$(X, \preceq)$ … poset $\rightsquigarrow$ order complex on X

simplices are chains

3) Nerve $\quad \mathcal{F} = \{\underbrace{F_1, \dots, F_n}_{\text{sets}, \neq \emptyset}\} \rightsquigarrow N(\mathcal{F})$ has vert. $[n] = \{1, \dots, n\}$

simplices $I \subseteq [n] : \bigcap_{i \in I} F_i \neq \emptyset$

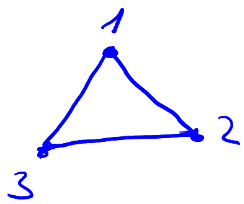
$\rightsquigarrow$



Nerve theorem $\quad K_1, \dots, K_n$ subcomplexes of finite simpl. comp. K

& $\bigcup_{i=1}^{n} K_i = K$. Suppose $\forall J \subseteq [n]$, $J \neq \emptyset$ $\quad \bigcap_{i \in J} |K_i|$ is empty or contractible

Then polyhedron of $N(|K_1|, \dots, |K_n|) \overset{\text{hom-equivalent}}{\simeq} |K|$

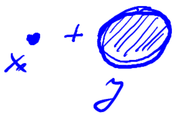


Alternatives to simpl. complexes

CW complexes building blocks are cells — balls of any dim.

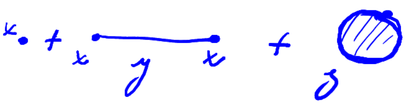
glued together s.t. n-dim. cell is attached along its boundary

to (n-1)-dim. part. of what is built already

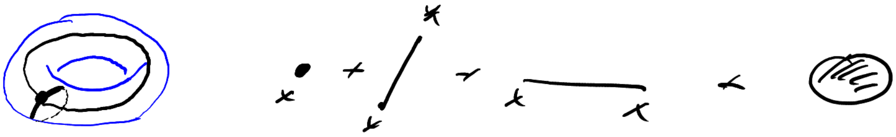


glued so that boundary of y = x

→ sphere



→ circle



simplicial sets glue simplices together

degenerate simplices ...

A simplicial set is a contravariant functor

$\Delta \to \mathrm{Set}.$

category of nonempty finite lin. ordered sets with mon. mappings as morphisms

category of sets

→ nice theor. prop.

→ algorithmic use

Non-embeddability

G is understood as simpl. complex

which graphs are planar? $f: |G| \to \mathbb{R}^2$ embedding (a homeomorphism to $f(|G|)$)

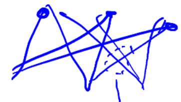
There are complete answers: G is planar

$\Updownarrow$

G has no subdivision of $K_{3,3}$ / K_5

$\Updownarrow$

G has no $K_{3,3}$-minor & no K_5-minor.



no!

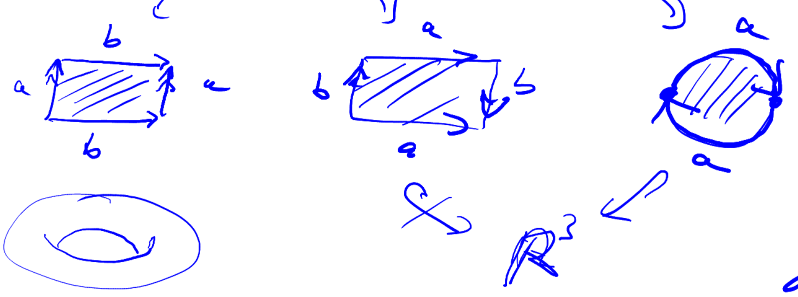
In general X, Y topol. spaces

Q: $\exists f: X \to Y$ embedding

X compact, Y Hausdorff $\Rightarrow$ we just need $f: X \to Y$ injective continuous

X: torus, Klein bottle, proj. plane $\qquad X \hookrightarrow \mathbb{R}^3$

1-1 mapping



no complete answers!

given X -- finite 4-dim. simpl. complex

$Y = \mathbb{R}^5$

it is undecidable whether $X \hookrightarrow Y$

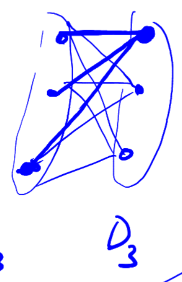
Recall: k-dimens. finite simpl. complex always embeds into $\mathbb{R}^{2k+1}$

Thm (Van Kampen & Flores) $\forall k \geq 1 \; \exists$ k-dim. fin. simpl. comp. $C_k \not\hookrightarrow \mathbb{R}^{2k}$

$D_3 = \vdots$ 3 isol. vertices

$$C_k = \underbrace{D_3 * D_3 * \dots * D_3}_{k+1}$$

$$C_2 \cong K_{3,3}$$



imagine cont. inj. $f: K \to \mathbb{R}^d$, want contradiction

$\forall x \neq y \quad f(x) \neq f(y)$

From $|K|$ we construct new spaces & mappings that will be antipodal $\longrightarrow$ Borsuk-Ulam thm

Abstract version of antipodality

concrete $x \longmapsto -x$

objects: $\mathbb{Z}_2$-space: (X, ν)

top. sp. X, $\nu : X \to X$, $\nu \circ \nu = id_X$

morphisms: $\mathbb{Z}_2$-map

$(X, \nu) \longrightarrow (Y, \omega)$

$f : X \xrightarrow{f} Y$ cont. s.t.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \nu \big\downarrow & \circ & \big\downarrow \omega \\ X & \xrightarrow[f]{} & Y \end{array}$$

$f \circ \nu = \omega \circ f$

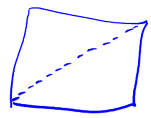
$\nu : X \to X$, $\omega : Y \to Y$ are homeomorphisms

(if $\nu(x) = -x$: $f(-x) = -f(x)$)

Deleted product

$K \rightsquigarrow |K|^2_\Delta$

$X \rightsquigarrow X^2_\Delta := \{ (x, y) : x, y \in X,\ x \neq y \}$



X^2_Δ has $\mathbb{Z}_2$-map $(x,y) \rightsquigarrow (y, x)$

$f : K \to \mathbb{R}^d \rightsquigarrow \tilde{f} : |K|^2_\Delta \to S^{d-1}$

$$\tilde{f}(x,y) = \frac{f(x) - f(y)}{\underbrace{\| f(x) - f(y) \|}_{\neq 0}}$$

& $\mathbb{Z}_2$-map