Class 6

Limit of a diagram

given: some objects & morphisms between them

we want: limit object T

morphisms $p_X : T \to X$

$p_Y : T \to Y$

$p_A : T \to A$

$$
\begin{tikzcd}
 & T' \arrow[dl, "p'_X"'] \arrow[dr, "p'_Y"] \arrow[d, "p'_A"] \arrow[ddd, bend left=60, "u"] & \\
X \arrow[r, "f"] & A & Y \arrow[l, "g"'] \\
 & & \\
 & T \arrow[uul, "p_X"] \arrow[uu, "p_A"'] \arrow[uur, "p_Y"'] &
\end{tikzcd}
$$

$\to$ pullback

s.t. 1) $f \circ p_X = p_A$ & $g \circ p_Y = p_A$

2) $\forall T', p'_X, p'_Y, p'_A$: $\exists$ unique $u : T' \to T$ s.t.

$p'_A = p_A \circ u$

$p'_X = p_X \circ u$

$p'_Y = p_Y \circ u$

In particular, limit of a diagram w/o any morphisms is ~~a~~ the product of objects in the diagram

Opposite category

Given some category $\mathcal{C}$ --- $Ob(\mathcal{C})$

$\forall X, Y \in Ob(\mathcal{C})$ --- $Hom_{\mathcal{C}}(X, Y)$

we create opposite category $\mathcal{C}^{op}$ by reversing all arrows & change the order in composition

$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

$Hom_{\mathcal{C}}(X, Y) \rightsquigarrow Hom_{\mathcal{C}^{op}}(Y, X)$

$$
\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[dl, "f"'] & \\
X & & Z \arrow[ul, "g"'] \arrow[ll, "f \circ g"]
\end{tikzcd}
$$

$\mathcal{C}$ $\qquad\qquad$ $\mathcal{C}^{op}$

limit of a diagram $\rightsquigarrow$ colimit

Simplicial complexes $\subseteq P(V)$

simplices (plural of simplex)

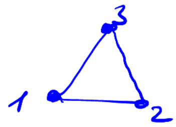
Df A simplicial complex is a system K of finite subsets of (possibly infinite) set V, s.t. $F' \subseteq F \in K \Rightarrow F' \in K$.

& $V = \bigcup K = \bigcup\{S : S \in K\}$ $\quad (V = V(K))$

points

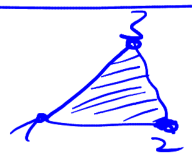
ground set

Ex



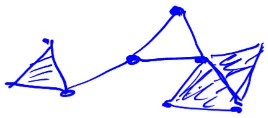
$V = \{1,2,3\}$

$K = \{12, 23, 13, 1, 2, 3, \emptyset\}$



$V = \{1,2,3\}$

$K' = K \cup \{123\}$



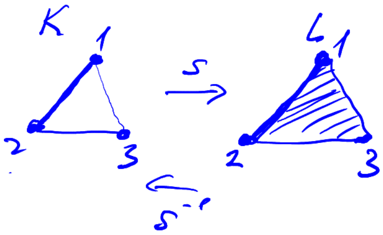
. . . . –

- $\dim K := \sup_{F \in K} (|F|-1)$ $\quad$ sup .... can be ∞ $\quad \dim K = 1, \ \dim L = 2$

- subcomplex L of K: $L \subseteq K$ & L is a complex
- induced subcomplex (by $U \subseteq V$): $L = \{F \in K : F \subseteq U\}$
- 1-dimensional complexes ⟿ (simple) graphs
- finite simplicial complex $\Leftrightarrow$ finite V .... correspond to compacts spaces

Df Simplicial map of K into L (simpl. complexes) is a map $s : V(K) \to V(L)$ s.t. $\forall F \in K : s(F) \in L$

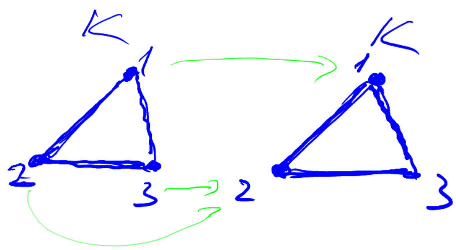
Isomorphism of K & L is a simpl. map $s : K \to L$ s.t. s is bijective & s^{-1} is simplicial



$s(i) = i$ $\quad \forall F \in K : s(F) \in L$ ✓

but not for s^{-1}

$s^{-1}(\{1,2,3\}) \notin K$



$s(\{1\}) = \{1\}$ $\qquad s(\{1,2\}) = \{1,2\}$

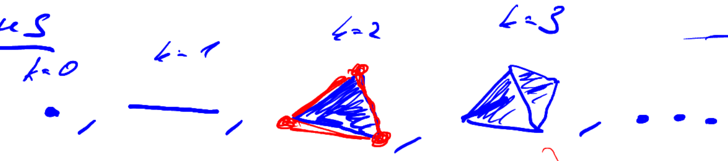
$s(\{2\}) = \{2\}$ $\qquad s(\{1,3\})$ "

. . . . $\qquad s(\{2,3\}) = \{2\} \in K$

different notion than graph homomorphism / alt. p.o.v.: our simpl. complexes .... graphs with loops at every vertex

- geometric realizations

geometric simplex: $k=0$ $\bullet$, $k=1$ —, $k=2$, $k=3$, ...



= convex hull of a set of affinely independent points in some $\mathbb{R}^n$

$k+1$ $\{p_0, p_1, \dots, p_k\}$

$\Updownarrow$

$p_1-p_0, \dots, p_k-p_0$ are linearly independent

1 empty set $\binom{4}{0}$
4 vert. $\binom{4}{1}$
6 edges $\binom{4}{2}$
4 faces $\binom{4}{3}$
1 itself $\binom{4}{4}$

faces of a simplex σ ... convex hulls of subsets

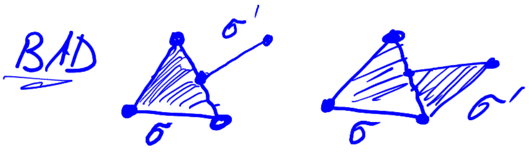
k-dim simplex has $\binom{k+1}{k}=k$ faces of dim. $k-1$

$\boxed{\text{Df}}$ Geometric simpl. complex is a coll. Δ of geometric simplices s.t.

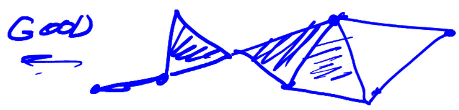
1) $\sigma \in \Delta$, σ' is a face of σ $\Rightarrow$ $\sigma' \in \Delta$

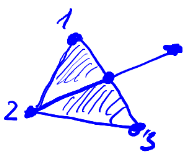
2) $\sigma, \sigma' \in \Delta$ $\Rightarrow$ $\sigma \cap \sigma'$ is a face of both σ & σ'

BAD



GOOD





$\boxed{\text{Observation}}$ Δ Geom. simpl. complex

$\Downarrow$

$K = K(\Delta)$ is a simpl. complex

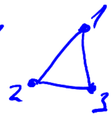
$V = V(\Delta)$ all 0-dim faces of $\sigma \in \Delta$

K - all faces of $\sigma \in \Delta$

Δ ... geometric realization of K

$\boxed{\text{Proposition}}$ $\forall$ finite K has a geometric realization; if $k = \dim K$, then the realization can be taken in $\mathbb{R}^{2k+1}$.

ex. $k=1$



$\notin \mathbb{R}^1$
$\subseteq \mathbb{R}^2$

$L :=$ nonplanar graph
$\Rightarrow$ no geom. real. in $\mathbb{R}^2$

$F = \{1,5\}$
$G = \{2,5\}$

Proof (sketch) K ... want to find placement of $V(K)$ s.t. no intersections

$\rho: V(K) \to \mathbb{R}^{2k+1}$ $\quad \forall F, G \in K$ $\quad \underbrace{\mathrm{conv}(\rho(F))}_{X} \cap \underbrace{\mathrm{conv}(\rho(G))}_{Y} = \mathrm{conv}(\rho(F \cap G))$

injective

it is enough if $\rho(F \cup G)$ is affinely independent set $\Rightarrow$ X, Y are faces of Z

$Z := \mathrm{conv}\, \rho(F \cup G)$ $\quad$ $X \cap Y$ is a face $= \mathrm{conv}(\rho(F) \cap \rho(G))$

$|F|, |G| \le k+1 \Rightarrow |\rho(F \cup G)| \le 2k+2$

So, we need the an n-point set X_n in $\mathbb{R}^{2k+1}$, where every $2k+2$ points are affinely independent

Then we def. $\rho : V(K) \to X_n$ arbitrarily (injective)

!!! Any $d+1$ distinct points on the moment curve $\{(t, t^2, \dots, t^d) : t \in \mathbb{R}\} \subseteq \mathbb{R}^d$ are aff. independent.

$\boxed{\text{Df}}$ Δ ... geom. simpl. complex; suppose all simplices in Δ are cont. in $\mathbb{R}^n$

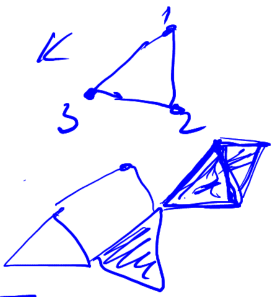
polyhedron of Δ := topol. space ... subspace of $\mathbb{R}^n$ induced by $\bigcup_{\sigma \in \Delta} \sigma$

A polyhedron of a finite simplicial complex K is the polyhedron of some geom. realization.

$|\Delta|$ $\qquad$ $|K|$

Note Another way to define $|K|$

$\forall F \in K$ we consider a separate simplex of dim $|F|-1$ + use disj. union of top. spaces + use quotient $/\sim$



2^3-1 simplices

$\boxed{\text{Goal}}$ Let K, L ... simpl. complexes

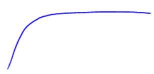
$s : V(K) \to V(L)$ simpl. map

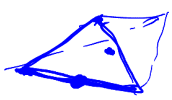
want: $|s| : |K| \to |L|$ ... cont. map. of polyhedra of K & of L

... there is a canonical way how to do it.

$\boxed{|s| : \Delta \to \Delta'}$

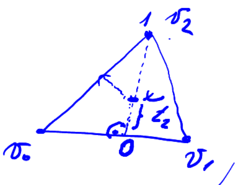
is it continuous? → Yes





σ ... geom. simplex = conv $\{v_0, \dots, v_k\}$

$x = \sum_{i=0}^{k} t_i v_i \qquad \left.\begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array}\right\}$ barycentric coordinates



fix geom. real. Δ ... of K

Δ' ... of L

pretend $s : V(\Delta) \to V(\Delta')$ points in an Eucl. space

For $x \in \Delta$ choose $\sigma \in \Delta$ s.t. $\sigma \ni x$ & σ has lowest dimension → unique

support of x

$x = \sum t_i v_i$

$|s|(x) := \sum_{i=0}^{k} t_i \, s(v_i) \in \Delta'$, $\quad s(v_i) \in V(\Delta')$

$\in \text{conv}\underbrace{\{s(v_0), \dots, s(v_k)\}}_{\in L,\ \text{simplex in } \Delta'}$