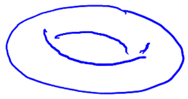
So far --- Set topology ... points & open sets ... local

Interlude ... Category theory

Move on --- Algebraic topology --- mostly --- global



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operations w. spaces
- quotient
- products
- disj. unions

Join $X * Y$ is obtained from $X \times Y \times [0,1]$

by factorization: each copy $X \times \{y\} \times \{0\}$ of X is collapsed to a point

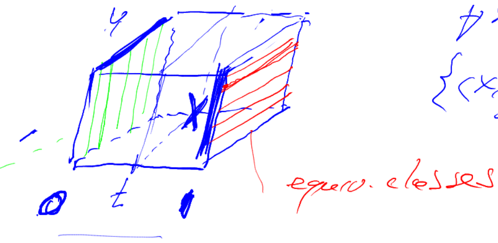
— " — $\{x\} \times Y \times \{1\}$ of Y — " —

Ex. $X = Y = [0,1]$

$X \times Y \times [0,1] = [0,1]^3$

$\forall x \in X,\ y \in Y,\ t \in (0,1)$

$\{(x,y,t)\}$ is a trivial eq. class



a tetrahedron = simplex

$\forall t \in (0,1)$

$X \times Y \times \{t\} \cong X \times Y$

$X * Y \supseteq X \times Y$

$\supseteq X$

$\supseteq Y$

geometric realizations of joins

$X \subseteq \mathbb{R}^m$, bounded

$Y \subseteq \mathbb{R}^n$ bdd

"non-intersecting"

embed $\mathbb{R}^m$ & $\mathbb{R}^n$ into $\mathbb{R}^{m+n+1}$ as skew affine subspaces

$\{x \in \mathbb{R}^{m+n+1} : x_{m+1} = x_{m+2} = \dots = 0\}$

$\{x_1 = \dots = x_m = 0,\ x_{m+1} = 1\}$

$X * Y$ is homeomorphic to

$\bigcup_{\substack{x \in X \\ y \in Y}} \overline{xy}$

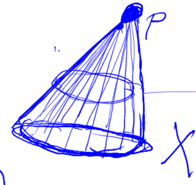
$\overline{xy}$ is line segment from x to y

this is a disj. union except the endpoints

Properties
- commutative (up to homeomorphism)
- associative (up to homeo) for compact Hausdorff spaces but not in general

Cone of a space X is $CX := X * \{p\}$

one point space



$X \times \{p\} \cong X$

$CX \cong X \times [0,1] / X \times \{1\}$

$\boxed{CX \text{ is always contractible}}$

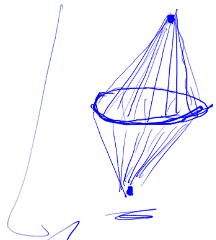
Suspension of X is $SX := X * S^0$

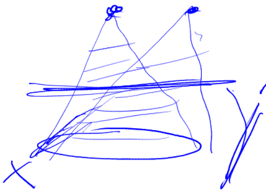
S^2 — sphere (2D object in $\mathbb{R}^3$)

S^1 — circle (1D obj. in $\mathbb{R}^2$)

S^0 ... $\{\pm 1\}$ in $\mathbb{R}^1$

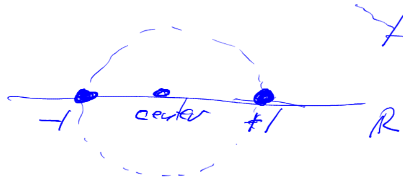
two points





$X \times S^0$

two disj. copies of X

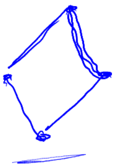


$\boxed{\text{suspension has holes}}$

$SS^n \cong S^{n+1}$

$n=1$

$n=0$



$\cong S^1$

Category theory

Category $\mathcal{C}$ consists of

① objects ..... any set / class

$Ob(\mathcal{C})$ ..... (class of) all topo. spaces Top

all groups Grp

graphs Gra

sets Set

⋮

② morphisms .... all mappings among objects of "the right type"

Set .... all mappings

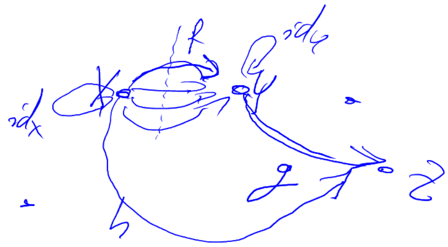
Grp ... all group homom.

Gra ... all graph homom.

Top ... all cont. maps

$Hom(X, Y)$ ..... all morph. from X to Y

→ then forget the rest ; So a category is



with compositions

axioms of cat. th. $\Rightarrow \exists h = gf$

Axioms 1) $\forall X, Y, U, V \in Ob(\mathcal{C})$: $Hom(X,Y) \cap Hom(U,V) = \emptyset$ (unless $X = U$, $Y = V$)

2) $\forall X \in Ob(\mathcal{C})$ $\exists\, id_X \in Hom(X, X)$

3) $\forall f \in Hom(X, Y)$ $\forall g \in Hom(Y, Z)$ $\exists h \in Hom(X, Z)$: $h = gf$

4) $f(gh) = (fg)h$

5) $f\, id_X = id_Y f = f$ $\quad \forall f \in Hom(X, Y)$

Df $f : X \to Y$ is a **monomorphism** iff it is "left-cancelable" ⊗ $\forall g_1, g_2 \in Hom(-, X)$

$fg_1 = fg_2 \Rightarrow g_1 = g_2$

— // — **epimorphism** iff it is right-cancellable

$g_1 f = g_2 f \Rightarrow g_1 = g_2$ $\quad \forall g_1, g_2 \in Hom(X, -)$

In Set monomorphism

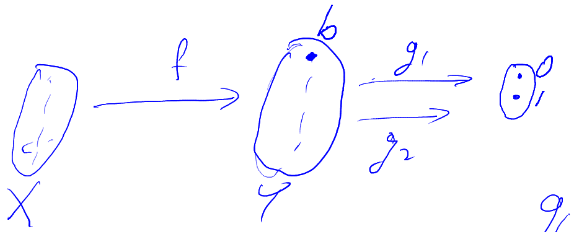
if $g_1(a) = x_1$

if f is monomorphism, THEN IN PART. ⊗ is true

for : $fg_1 = fg_2 \Rightarrow g_1 = g_2$ ⓘ THEREFORE f is injective

$f(x_1) = f(x_2)$ $\quad\quad x_1 = x_2$

In Set Let f be an epimorphism



g_1 maps everything in Y to 0

g_2 — " — to 0 except $g_2(b)=1$

$g_1 \neq g_2 \Rightarrow g_1 f \neq g_2 f$

$g_1 f$ = everything in Y to 0

$\Rightarrow f(x)=b$ for some $x \in X$

THEN f is surjective

Df $f: X \to Y$ is an isomorphism if $\exists g: Y \to X$ s.t. $fg = id_Y$, $gf = id_X$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Thm isomorphism $\Rightarrow$ both mono- & epimorphism

$\not\Leftarrow$ not true in general

Products Given $X, Y \in Ob(\mathcal{C})$, we define their product P, $p_X: P \to X$, $p_Y: P \to Y$

$$\begin{tikzcd} & P \arrow[dl, "p_X"'] \arrow[dr, "p_Y \in Hom(P,Y)"] \arrow[d, shift right, "g"'] & \\ X & P' \arrow[l, "p'_X"] \arrow[r, "p'_Y"'] \arrow[u, shift right, "\exists! f"'] & Y \end{tikzcd}$$

Such that $\forall P', p'_X, p'_Y$ $\exists$ unique $f: P' \to P$

$p_X f = p'_X$ & $p_Y f = p'_Y$

this diagram commutes

→ If P exists, then it is unique up to isomorphism
Spose both P & P' sat. the def. Then $\exists f$ & $\exists g$

True Thm, not proved here

what? $gf = id_P$?

$$p'_X(gf) = (p'_X g) f = p_X f = p'_X$$

As P', p'_X, p'_Y is product

In Set $P = X \times Y$ is a product

$p_X: (x,y) \mapsto x$

$p_Y: (x,y) \mapsto y$

$$\begin{tikzcd} & P = X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & P' \arrow[l, "p'_X"] \arrow[r, "p'_Y"'] \arrow[u, "f"] & Y \end{tikzcd}$$

$a \in P'$ $f(a) = (p'_X(a), p'_Y(a))$