Planar point sets determine many pairwise crossing segments

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Theorem 1 (main result)**.** *Any set V of n points in general position in the plane determines at least $\frac{n}{2^{O(\sqrt{\log n})}}$ pairwise crossing segments.*

Denote the open half-plane to the left of the oriented line xy by $l(xy)$. For a non-empty set B in the plane and any two points in the plane, x and y, write $x <_B y$ if B is contained in $l(xy)$.

Lemma 2. *Let A and B be nonempty sets in the plane such that their convex hulls are disjoint. Then $<_B$ defines a partial order on A, in which two distinct points x and y are incomparable if and only if the line through x and y intersects* $\operatorname{conv}(B)$*. Further, if $x <_B y$ for $x, y \in A$ and $z <_A t$ for $z, t \in B$, then the segments xz and yt cross.*

Denote by $\iota(P, <)$ the number of incomparable elements in the partial order $(P, <)$.

Definition 1. Two point sets $A, B \subset \mathbb{R}^2$ are called *separated* if $\operatorname{conv}(A) \cap \operatorname{conv}(B) = \emptyset$. Two separated m-element sets, A and B, are said to form an *ε-avoiding pair* for some $\varepsilon \geq 0$ if

$$\iota(A, <_B) + \iota(B, <_A) \leq \varepsilon m^2.$$

For simplicity, a 0-avoiding pair is called *avoiding.*

Theorem 1 can be proven recursively. The initial step is the following lemma.

Lemma 3. *Given n points in a plane in general position we can find an ε-avoiding pair of size $\Omega(\varepsilon^4 n)$.*

Or, in other words, there exists an absolute constant $c > 0$ such that for any integer m and any real $\varepsilon > 0$, every point set in general position with least $\frac{cm}{\varepsilon^4}$ points has two separated m-element sets that form an ε-avoiding pair.

And the recursive step is given by the next lemma.

Lemma 4. *Given ε-avoiding pair of size m we can find $\frac{k}{8}$ pairwise crossing "super-edges" each on $\frac{m}{k} + \frac{m}{k}$ vertices, each forming 8ε-avoiding pair.*

Or, in other words, let k, m and $t = 8$ be positive integers and set $\varepsilon = \frac{1}{32t^2k}$. Let A and B be sets of points such that $|A| = |B| = (t+1)km$ and $\{A, B\}$ is $\frac{\varepsilon}{t}$-avoiding pair.

Then we can find pairwise disjoint m-element subsets $A_1, A_2, \ldots, A_k \subset A$ and $B_1, \ldots, B_k \subset B$ such that for every $1 \leq i \leq k$, $\{A_i, B_i\}$ is ε-avoiding pair and, for $1 \leq i < j \leq k$, all segments between A_i, B_i cross every segment between A_j, B_j.

1 Proof of Lemma 3

Lemma 5 (Matoušek)**.** *For any n-element point set V in general position in the plane and for any $\varepsilon > 0$, we can find $O\left(\frac{1}{\varepsilon^2}\right)$ lines such that the zone of any other line within their arrangement contains at most εn points of V.*

By applying this lemma we can get arrangement of $r = \frac{1}{\varepsilon^2}$ lines. We get approximately r^2 cells. Split cells into smaller clusters of size m. Some vertices are on the lines and some in cells smaller than m but we still have $\geq \frac{n^2}{2m^2} - \frac{2r^2n}{m}$ pairs of clusters containing m points.

Compute the number of not ε-avoiding pairs clusters, and conclude that some ε-avoiding pair exists.

2 Proof of Lemma 4

Lemma 6. *Let n and k be positive integers, and $(P, <)$ be a poset with $|P| > nk$ and $\iota(P, <) \leq \frac{(|P|-nk)^2}{16k}$. Then one can choose suitable n-element subsets $A_1, A_2, \ldots, A_k$ of P with $A_i < A_j$ for all $i < j$.*

Apply Lemma 6 to find suitable subsets $C_i \subseteq A$ and $D_i \subseteq B$ for $1 \leq i \leq tk$ such that $|C_i| = |D_i| = m$ for all i, and $C_i <_B C_j$, $D_i <_A D_j$ for all $i < j$.

By Lemma 2 it suffices to find a collection of ε-avoiding pairs $\{(C_a, D_b)\}$ such that $b = b(a)$ is a monotone increasing function of a.

Lemma 7 (Mirsky)**.** *Let $(P, <)$ be a finite poset. Then the length of the largest chain in $(P, <)$ is equal to the size k of its smallest possible decomposition into antichains.*

Claim 8. *If points x and y in A are incomparable in $<_B$, they are still comparable with respect to all but at most one ordering $<_{D_b}$, $1 \leq b \leq tk$.*

Using this claim we can bound the number of not ε-avoiding pairs and show that at least $6tk^2$ ε-avoiding pairs exist. Considering partial order defined by $\{C_a, D_b\} < \{C'_a, D'_b\}$ if $a < a'$ and $b < b'$, and using Mirsky theorem we get the suitable collection as a chain of this partial order.

3 Proof of Theorem 1

Lemma 9. *Let s be a positive integer and set $K = 8^{\binom{s}{2}}$, $M = 9^s K$, $\varepsilon = 2^{-3s-11}$. Suppose the M-element point sets A and B form a ε-avoiding pair and $A \cup B$ is in general position. Then we can find K pairwise crossing segments, each connecting a point of A to a point of B.*

Prove this lemma by induction on s using Lemma 4. Then use this lemma and Lemma 3 to prove the theorem.