KEMPE EQUIVALENCE CLASSES OF CUBIC GRAPHS EMBEDDED ON THE PROJECTIVE PLANE (KENTA OZEKI)

presented by

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In the following, G always denotes a cubic, i.e. 3-regular, graph and φ a 3-edge-coloring of G.

A *Kempe switch* of a 3-edge-coloring of a cubic graph G on a bicolored cycle C swaps the colors on C and gives rise to a new 3-edge-coloring of G. Two 3-edge-colorings of G are Kempe equivalent if they can be obtained from each other by a sequence of Kempe switches.

The 3-edge-colorability problem for cubic graphs is NP-complete, while every cubic bipartite graph admits a 3-edge-coloring.

A surface F^2 is a connected compact 2-dimensional manifold without boundary. A closed curve γ on F^2 is essential if γ does not bound a 2-cell region on F^2. Otherwise, γ is contractible. A closed curve γ on F^2 is one-sided if a tubular neighborhood of γ is a Möbius strip; otherwise it is two-sided. When the surface F^2 is the projective plane, a closed curve is essential if and only if it is one-sided.

Lemma 1. *Let γ_1 and γ_2 be two closed curves on the projective plane. Then γ_1 and γ_2 are both essential (i.e. one-sided) if and only if they intersect transversally an odd number of times.*

A triangulation of a surface F^2 is a graph embedded on F^2 with each face triangular. A facial walk in a graph embedded on a surface is the boundary walk of some face.

For a graph G embedded on a surface, the dual of G is denoted by G^* . For $S \subseteq E(G)$, we denote by S^* the set of dual edges e^* in G^* taken over all edges e in S.

A *2-factor* is a spanning subgraph in which every vertex has degree exactly 2.

Lemma 2. *Let G be a graph embedded on the projective plane, and let T be a 2-factor of G. Then T satisfies exactly one of the following: (I) All cycles in T are contractible, and T is 2-face-colorable. (II) T contains exactly one essential cycle, and T is not 2-face-colorable*

We consider φ as a mapping from E(G) to $Z_2 \times Z_2 - \{(0,0)\}$. It produces a mapping $f_\varphi : V(G^*) \to Z_2 \times Z_2$ of dual triangulation G^* of G as follows. First, fix a vertex of G^* and color it by (0, 0), and then we extend the colors using the following rule: Let h be already colored and h_0 is adjacent to h not colored yet with e^* the edge connecting them in G^* . Then we color h_0 so that $f_\varphi(h0) = f_\varphi(h) + \varphi(e)$, where $+$ means the sum on $Z_2 \times Z_2$. It has been shown that f_φ is well-defined and indeed a 4-vertex-coloring of G^*. Given an 4-edge-coloring f of G^* we can obtain a 3-edge coloring φ_f of G by using opposite rules.

Let $X \subseteq E(G)$ and M be a perfect matching of G. Then we define the *type* of X as $\sigma_X(M) \equiv |X| - |X \cap M| (\text{mod } 2)$ with $\sigma_X(M) \in \{0,1\}$.

Let a, b, c denote the colors of φ and M_x stands for the edges colored by a color x for $x \in \{a, b, c\}$. Note that each of M_a, M_b, M_c is a perfect matching in G. We define the type with respect to X, denoted by $\sigma_X(\varphi)$, of φ as the triple of $\sigma_X(M_x)$ for $x \in a, b, c$.

Proposition 3. *For a 3-edge-coloring φ of a cubic graph G and a subset X of $E(G)$, we have $\sigma_X(\varphi) = (0,0,0), (1,1,0), (1,0,1),$ or $(0,1,1)$.*

Theorem 4. *Let G be a cubic graph embedded on a surface. A 3-edge-coloring φ of G produces a 4-vertex-coloring of the dual G^* if and only if $\sigma_D(\varphi) = (0,0,0)$ for every essential cycle D^* in G^* .*

Theorem 5. *If two 3-edge-colorings of a cubic graph embedded on the projective plane are Kempe equivalent, then they have the same type.*

Given φ_1 and φ_2 3-edge-colorings, we can define their *signature* as follows. For a vertex v and e_1, e_2, e_3 edges incident with v, π_v is a permutation on three elements such that $\pi_v \varphi_1(e_i) = \varphi_2(e_i)$ for any $1 \leq i \leq 3$. The signature between φ_1 and φ_2 is then $\text{sign}(\varphi_1, \varphi_2) = \Pi_{v \in V(G)} \text{sign}(\pi_v)$.

An edge e is said to be *singular* (with respect to φ) if for a face h containing e, the two edges incident with h and adjacent to e have the same color by φ: Otherwise, e is said to be non-singular. We denote $NS_x(\varphi)$ the non-singular edges of color x in coloring φ.

The *signature* of a 3-edge-coloring φ with respect to essential cycle D^* of G^*, denoted by $\text{sign}_D(\varphi)$ is defined as follows. After deleting the edges of D, we obtain a spanning subgraph G' contained on a disc. On this disc, u, v have the same consistent clockwise rotation if and only if $uv \notin D$. If colors a, b, c occur on the edges incident with v along this rotation, we let $\text{sign}(\pi_v) = +1$. Otherwise, $\text{sign}(\pi_v) = -1$.

$$\text{sign}_D(\varphi) = \Pi_{v \in V(G)} \text{sign}(\pi_v).$$

Theorem 6. *Every cubic bipartite graph G embedded on the projective plane has an odd 3-edge-coloring. Moreover, it has an even 3-edge-coloring if and only if the dual triangulation G^* has chromatic number at most 4.*

Theorem 7. *Let G be a cubic bipartite graph embedded on the projective plane, and let D^* be an essential cycle in G^* . Then G admits a perfect matching M with $\sigma_D(M) = 1$.*

A *canonical 3-edge-coloring* of a cubic graph embedded on a surface is a 3-edge-coloring such that for each color x, either $NS_x(\varphi)^*$ consists of only essential cycles or is the empty set.

Lemma 8. *Let G be a cubic even map on a surface. For every 3-edge-coloring φ of G, there exists a canonical 3-edge-coloring φ_0 such that φ_0 is Kempe equivalent to φ and for every cycle D^* in G^* , $\sigma_D(\varphi) = (0,0,0)$ if and only if $\sigma_D(\varphi_0) = (0,0,0)$.*

For H an even map on the projective plane, it's *face subdivision* $FS(H)$ is an Eulerian triangulation of H created by adding a vertex into each face and connecting it to the vertices of the corresponding facial walk. For an Eulerian triangulation K, we say that vertex set U is a color factor of K, if there exists an even map H such that $K = FS(H)$ and $U = V(K) - V(H)$.

Theorem 9. *Let G be a cubic bipartite graph embedded on the projective plane. Then any two odd 3-edge-colorings in G are Kempe equivalent.*

Theorem 10. *All 3-edge-colorings of a cubic bipartite projective-planar graph G are pairwise Kempe equivalent if and only if G has an embedding in the projective plane such that the chromatic number of the dual triangulation G^* is at least 5.*

Corollary 11. *Let G be a 3-edge-colorable cubic graph embedded on the projective plane. If the dual G^* is not 4-vertex-colorable, then G is 3-list- edge-colorable.*