Breaking the degeneracy barrier for coloring graphs with no K_t minor

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Conjecture (Hadwiger). *For every integer $t \geq 1$, every graph with no K_t minor is $(t-1)$-colorable.*

Theorem 1.3. *(Kostochka, Thomason) Every graph with no K_t minor is $\mathcal{O}(t\sqrt{\log t})$-degenerate (and colorable).*

Theorem 1.4. *For every $\beta > \frac{1}{4}$, every graph with no K_t minor is $\mathcal{O}(t(\log t)^{\beta})$-colorable.*

Theorem 2.4. *For every $\delta > 0$ there exists $C = C_{2.4}(\delta) > 0$ such that for every $D > 0$ the following holds. Let G be a graph, $d(G) \geq C$, $s = D/d(G)$. Then G contains at least one of the following:*

(i) a minor J, $d(J) > D$

(ii) a subgraph H, $v(H) \leq Cs^{\delta}D^2/d(G)$ and $d(H) \geq d(G)/(Cs^{\delta})$

Theorem 2.6. *There exists $C_{2.6} > 1$ satisfying the following. Let G be a graph with $\kappa(G) \geq Ct(\log t)^{\frac{1}{4}}$, and let $r \geq \sqrt{\log t}/2$. If there exist vertex-disjoint subgraphs $H_1, \ldots, H_r$ of G such that $d(H_i) \geq Ct(\log t)^{\frac{1}{4}}$ for each i, then G has a K_t minor.*

Theorem 4.2. *Let $l \geq 2$ be an integer, and let $\epsilon_0 \in (0, \frac{1}{2l})$ and $d_0 \geq 1/\epsilon_0$ be real. Let $G = (A, B)$ be a bipartite graph such that $|A| > l|B|$ and every vertex in A has at least d_0 neighbors in B. Then G contains at least one of the following:*

(i) a subgraph H with $v(H) \leq 3d_0$ and $e(H) \geq \epsilon_0^2 d_0^2/2$

(ii) an $(l+1)$-bounded minor G' with $d(G') \geq \frac{l}{2}(1 - 2l\epsilon_0)d_0$

Theorem 4.4. *Let $k \geq l \geq 2$ be integers. Let $\epsilon \in (0, \frac{1}{4k})$. Let G be a graph with $d = d(G) \geq 1/\epsilon$. Then G contains at least one of the following:*

(i) a subgraph H with $v(H) \leq 3k^3 d$ and $e(H) \geq \epsilon^2 d^2/2$

(i) a bipartite subgraph $H = (X, Y)$ with $|X| > l|Y|$ such that every vertex in X has at least $(1 - 2k\epsilon)d$ neighbors in Y

(iii) an k-bounded minor G' with $d(G') \geq \frac{k}{8l}(1 - 2k\epsilon)d$

Lemma 3.2. *Let G be a graph with $d = d(G) \geq 3$. Then G has a minor H such that $v(H) \leq d + 2$ and $2\delta(H) \geq v(H) + 0.3d - 2$.*

Lemma 3.3. *Let $n \geq 0$, $k \geq 2$ and $h \geq n + 3k/2$ be integers. Let G be a graph with $\kappa(G) \geq k$ containing vertex-disjoint non-empty connected subgraphs $C_1, \dots, C_h$ such that each of them is non-adjacent to at most n others. Let $S = \{s_1, \dots, s_k\} \subset V(G)$. Then G contains vertex-disjoint non-empty connected subgraphs $D_1, \dots, D_m$ where $m = h\lfloor k/2 \rfloor$ such that $s_i \in V(D_i)$ for each $i \in [k]$ and every element of $\{D_1, \dots, D_m\}$ is non-adjacent to at most n subgraphs among $D_{k+1}, \dots, D_m$.*

Lemma 3.4. *There exists $C = C_{3.4} > 0$ satisfying the following. Let s be a positive integer, let G be a graph with $\kappa(G) \geq Cs$, and let $S_1, \dots, S_k$ be non-empty disjoint subsets of $V(G)$ such that $\sum_{i=1}^{k} |S_i| \leq s$. Then there exist vertex-disjoint connected subgraphs $C_1, \dots, C_k$ of G such that $S_i \subset V(C_i)$ for every $i \in [k]$.*

Lemma 3.5. *There exists $C = C_{3.5} > 0$ satisfying the following. Let G be a graph, let $l \geq s \geq 2$ be positive integers. Let $s_1, \dots, s_l, t_1, \dots, t_l, r_1, \dots, r_s \in V(G)$ be distinct, except possibly $s_i = t_i$ for some $i \in [l]$. If*

$$\kappa(G) \geq Cmax\{l, s\sqrt{\log s}\}$$

then there exists a K_s model $\mathcal{M}$ in G rooted at $\{r_1, \dots, r_s\}$ and an $(s_i, t_i)_{i \in [l]}$-linkage $\mathcal{P}$ in G such that $\mathcal{M}$ and $\mathcal{P}$ are vertex-disjoint.