Solution to the Ringel's circle problem

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Definition 1 The tangency graph of a set of circles $\mathcal{C}$ is $G(\mathcal{C}) = (\mathcal{C}, \{\{C_1, C_2\} \mid C_1, C_2 \in \mathcal{C}, |C_1 \cap C_2| = 1\})$

Chromatic number of a collection of circles is chromatic number of its tangency graph.

Problem 1 *What is the maximum chromatic number of a general constellation(no three circles are tangent in a same point) of circles in the plane?*

Theorem 1
There exists constellations of circles in the plane with arbitrarily large chromatic number (and girth).

If instead of using tangency of circles in $G(\mathcal{C})$ to define edges we can define edges as pairs of circles that intersect in a given angle θ, we call this graph θ-graph of $\mathcal{C}$.

Theorem 2
For every $\theta \in [0, \frac{\pi}{2}]$, there exists a θ-graph of circles (no two concentric) in the plane with arbitrarily large chromatic number (and girth).

Proof of Theorem 2

Definition 2 Homothetic map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map of the form $f(x) = sx + c$ for some constants $s > 0$ and c.

Theorem 3 (Gallai's Theorem and generalization of Prömel and Voight)
For every finite set $T \subset \mathbb{R}^d$, (and every $g \geq 3$), there exists a finite set $X \subset \mathbb{R}^d$ such that every k-coloring of X contains a monochromatic homothetic copy of T (, and no fewer than g homothetic copies of T in X forms a cycle on X).

Gallai's theorem can be proved from Hales-Jewett Theorem.

For $n \in \mathbb{N}$ and a finite set T, a subset L of the n-dimensional hypercube T^n is called a combinatorial line if there exist a non-empty set of indices $I = \{i_1, \dots, i_k\} \subseteq [n]$ and a choice of $t_i^* \in T$ for every $i \in [n] \setminus I$ such that

$$L = \left\{(t_1, \dots, t_n) \subset T^n \mid t_{i_1} = \dots t_{i_k} \text{ and } t_j = t_j^* \ \forall j \in [n] \setminus I\right\}$$

Theorem 4 (Hales-Jewett Theorem and generalization of Prömel and Voight)
For any finite T and $k \in \mathbb{N}$ (and $g \in \mathbb{N}$), there exist $n \in N$ such that every k-coloring of T^n contains a monochromatic combinatorial line.

(And more generally additionally exists $H \subseteq T^n$ such that every k-coloring of H contains a monochromatic combinatorial line and no tuple of fewer than g combinatorial lines in H form a cycle.)

Proof of Theorem 1

We say that a family $\mathcal{F}$ of $2d$-variate real polynomials respects a set $X \subset \mathbb{R}^d$ if $f(p,q) \neq 0$ for all $f \in \mathcal{F}$ and all pairs of distinct points $p, q \in X$.

Theorem 5
For every finite set $T \subset \mathbb{R}^d$, and every countable family $\mathcal{F}$ of $2d$-variate real polynomials that respects T, there exists a finite set $X \subset \mathbb{R}^d$, such that every k-coloring of X contains a monochromatic homothetic copy of T and $\mathcal{F}$ respects X.

Theorem 6
For any integers ($g > 3$) and $k > 1$, there exists a collection of circles $\mathcal{C}$, no two concentric and no two internally tangent, such that the tangency graph $G(\mathcal{C})$ has (girth at least g) and chromatic number at least k.