Tietze Theorem on Extension.

(for those who would like to know how to prove it)

1. From the Text we will need the concept of uniform convergence of mappings between metric spaces (Chapter XVIII, Section 1):
$f_n : X \to Y$ *uniformly coverge* to $f : X \to Y$, notation

$$f_n \rightrightarrows f,$$

if

$$\forall \varepsilon \exists n_0,\ n \geq n_0 \ \Rightarrow\ \forall x, |f_n(x) - f(x)| < \varepsilon$$

(note that n_0 depends on ε only, **not** on x unlike in the *pointwise convergence* $f_n \to f$ where we just assume that $\lim_n f_n(x) = f(x)$ for all x).
Unlike in pointwise convergence, one has that

if $f_n \rightrightarrows f$ and if all f_n are continuous then f is continuous.

(Text XVIII,1.3, very easy.)
Needless to say, a series of functions

$$f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots$$

uniformly converges if the sequence $(\sum_{k=1}^{n} f_k(x))_n$ uniformly converges.

2. Tietze Theorem for functions with values in a compact interval.
Given: X a metric space, $Y \subseteq X$ closed, $f : Y \to \langle -1, 1\rangle$ a continuous map.

Goal: A continuous $g : X \to \langle -1, 1\rangle$ with $g|Y = f$.

Expedient functions to be used: $A, B \subseteq X$ disjoint closed, $\alpha, \beta \in \langle -1, 1\rangle$. Set

$$\phi\{A, B; \alpha, \beta\}(x) = \alpha + (\beta - \alpha)\frac{d(x, A)}{d(x, A) + d(x, B)}$$

(d is the distance in X). Note that because of A, B being disjoint closed, $d(x, A) + d(x, B) \neq 0$ for all x ($d(x, A) + d(x, B) = 0$ yields $x \in \overline{A} \cap \overline{B} = A \cap B$) and hence $\phi\{A, B; \alpha, \beta\}$ is a continuous function. For $\phi = \phi\{A, B; \alpha, \beta\}$ one has

$$\phi[A] \subseteq \{\alpha\}, \phi[B] \subseteq \{\beta\} \text{ and all } \phi(x) \text{ are between } \alpha \text{ and } \beta.$$

Construction: Set $f_0 = f$, $A_1 = f^{-1}[\langle -1, -\frac{1}{3}\rangle]$, $B_1 = f^{-1}[\langle \frac{1}{3}, 1\rangle]$, $\phi_1 = \phi\{A_1, B_1; -\frac{1}{3}, \frac{1}{3}\}$ and $f_1(x) = f(x) - \phi_1(x)$ for $x \in Y$. Then

$$|\phi_1(x)| \leq \frac{1}{3} \quad \text{and} \quad |f_1(x)| \leq \frac{2}{3}.$$

Now suppose we already have continuous

$$\phi_1, \dots, \phi_n \text{ on } X \quad \text{and} \quad f = f_0, f_1, \dots, f_n \text{ on } Y \qquad (*)$$

such that for all $k \leq n$

$$|f_k(x)| \leq \left(\frac{2}{3}\right)^k \quad \text{and} \quad |\phi_k(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}. \qquad (**)$$

Now set
$A_{n+1} = f_n^{-1}[\langle -\left(\frac{2}{3}\right)^n, -\frac{1}{3}\left(\frac{2}{3}\right)^n\rangle]$, $B_{n+1} = f_n^{-1}[\langle \frac{1}{3}\left(\frac{2}{3}\right)^n, \left(\frac{2}{3}\right)^n\rangle]$,
$\phi_{n+1} = \phi\{A_{n+1}, B_{n+1}; -\frac{1}{3}\left(\frac{2}{3}\right)^n, \frac{1}{3}\left(\frac{2}{3}\right)^n\}$,
and $f_{n+1}(x) = f_n(x) - \phi_{n+1}(x)$ for $x \in Y$
(we are doing with f_n what we have done with $f = f_0$ above). Thus, the ϕ_k and f_k from $(*)$ satisfying $(**)$ are extended one more step.

The desired function g: For $x \in X$ set

$$g(x) = \sum_{n=1}^{\infty} \phi_n(x).$$

This defines a function $X \to \langle -1, 1\rangle$ (as $|g(x)| \leq \sum_{n=1}^{\infty} |\phi_n(x)| \leq \sum_{n=1}^{\infty} \phi_n(x) \leq \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = \frac{1}{3}3 = 1$) and this function is continuous because the series (that is, the sequence $(\sum_{k=1}^{n} \phi_n(x))_n$) obviously converges uniformly.
Finally, for $x \in Y$ we have $f_k(x) = f_{k+1}(x) + \phi_{k+1}(x)$ and hence

$$f(x) = f_1(x) + \phi_1(x) = f_2(x) + \phi_2(x) + \phi_1(x) = \cdots$$
$$\cdots = f_n(x) + \sum_{k=1}^{n} \phi_n(x)$$

and since $\lim_n f_n(x) = 0$, $f(x) = \lim_n \sum_{k=1}^{n} \phi_n(x) = g(x)$.

Summarizing, and taking into account that each non-trivial compact interval is homeomorphic with $\langle -1, 1\rangle$, we obtain

Theorem. (Tietze) *Let Y be a closed subspace of a metric space X, let J be a compact interval and let $f : Y \to J$ be continuous. Then there is a continuous $g : X \to J$ such that $g|Y = f$.*

More explicitly, let Y be closed in a metric space X. Then each real function f on Y such that $a \leq f(x) \leq b$ for all x can be extended to an equally bounded continuous g on X.