Numbers.

Natural numbers

$\mathbb{N}$ 0, successor, induction
arithmetic: addition, multiplication, 1=0'

$$a + 0 = a \qquad a \cdot 1 = a$$
$$(a+b)+c = a+(b+c) \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$a + b = b + a \qquad a \cdot b = b \cdot a$$
$$(a+b)c = ac + bc$$

Morover:
Order $a \leq b \quad (\exists x, a + x = b)$

$$a \leq a \quad \text{and} \quad a \leq b \;\&\; b \leq a \text{ only if } a = b$$
$$a \leq b \text{ and } b \leq c \;\Rightarrow\; a \leq c$$
$$\forall a, b, \text{ either } a \leq b \text{ or } b \leq a$$
$$a \leq b \;\Rightarrow\; a + c = b + c$$
$$a \leq b \;\Rightarrow\; a \cdot c = b \cdot c$$

Peano axioms.

$n \neq 0$ is m' of precisely one m

0 is not a successor

If $A(0)$ and $A(n) \Rightarrow A(n')$ then $\forall n A(n)$

($A(n) \equiv$ "*Statement A holds for n*")

$n + 0 = 0, \quad n + m' = (n + m)'$

$n0 = 0, \quad nm' = nm + n$

Examples: 1. *Associativity of addition*:

$(m+n)+0 = m+n = (m+0)+n, \quad m+(n+p') = m+(n+p)' = (m+(n+p))' = ((m+n)+p)' = (m+n)+p'$

2. $0 + n = n$:

$0 + 0 = 0, \quad 0 + n' = (0 + n)' = n'$

3. *Set* $1 = 0'$. *Then* $n' = n + 1$ *and* $n' = 1 + n$:

$0' = 1 = 1 + 0 = 0 + 1, \quad n' = (n + 0)' = n + 0' = n + 1, \quad 1 + n' = (1 + n)' = (n')'$

4. *Commutativity of addition*:

$m+n' = (m+n)' = (n+m)' = n+m' = n+(1+m) = (n + 1) + m = n' + m$

5. *Distributivity*:

$(m+n)p' = (m+n)p+m+n = mp+np+m+n = (mp + m) + (np + n) = mp' + np'$

Integers

$\mathbb{Z}$: arithmetic: addition, 0, subtraction, multiplication,
1

$$a + 0 = a \qquad a1 = a$$

$$(a+b)+c = a+(b+c) \qquad (ab)c = a(bc)$$

$$a + b = b + a \qquad ab = ba$$

$$(a+b)c = ac + bc$$

$$\forall a \exists b,\ a + b = 0 \quad \text{(NEW)}$$

Morover: order again

$a \leq a$ and $a \leq b$ & $b \leq a$ only if $a = b$

$a \leq b$ and $b \leq c \ \Rightarrow\ a \leq c$

$\forall a, b,$ either $a \leq b$ or $b \leq a$

$a \leq b \ \Rightarrow\ a + c = b + c$

$a \leq b$ and $0 \leq c \ \Rightarrow\ ac \leq bc$ (MODIFIED)

Rationals

$\mathbb{Q}$: arithmetic: addition, 0, subtraction, multiplication, 1, division

Order

$a + 0 = a$ $\qquad$ $a1 = a$

$(a+b)+c = a+(b+c)$ $\qquad$ $(ab)c = a(bc)$

$a + b = b + a$ $\qquad$ $ab = ba$

$(a+b)c = ac + bc$

$\forall a \exists b,\ a + b = 0$

$\forall a \neq 0 \exists b,\ ab = 1$ (NEW)

$a \le a$ and $a \le b$ & $b \le a$ only if $a = b$

$a \le b$ and $b \le c \Rightarrow a \le c$

$\forall a, b,$ either $a \le b$ or $b \le a$

$a \le b \Rightarrow a + c = b + c$

$a \le b$ and $0 \le c \Rightarrow ac \le bc$

(ORDERED FIELD)

Reals

$\mathbb{R}$: arithmetic: addition, 0, subtraction, multiplication, 1, division

Order

$$a + 0 = a \qquad a1 = a$$
$$(a+b)+c = a+(b+c) \qquad (ab)c = a(bc)$$
$$a + b = b + a \qquad ab = ba$$
$$(a+b)c = ac + bc$$
$$\forall a \exists b,\ a + b = 0$$
$$\forall a \neq 0 \exists b,\ ab = 1$$

$$a \leq a \quad \text{and} \quad a \leq b \;\&\; b \leq a \text{ only if } a = b$$
$$a \leq b \text{ and } b \leq c \ \Rightarrow\ a \leq c$$
$$\forall a, b,\ \text{either } a \leq b \text{ or } b \leq a$$
$$a \leq b \ \Rightarrow\ a + c = b + c$$
$$a \leq b \text{ and } 0 \leq c \ \Rightarrow\ ac \leq bc$$

and each non-void bounded subset has a supremum. (NEW)

(COMPLETELY ORDERED FIELD)

Notes.

1. We work with reals assuming exactly the properties just listed.

2. Complex numbers constitute a field, but cannot be (linearly) ordered.

3. Reals are given more structure, namely distance $|x - y|$, making them to Euclidean line.

4. The term *complete* has two senses. In order theory it refers to the existence of suprema of every subset of an ordered set. In metric spaces it refers to convergence of all Cauchy sequences. Thus for instance every Euclidean space (including the complex plane) is complete in this second sense without being ordered at all.

5. Reals are complete in both senses, that is, strictly speaking they are complete in the metric sense and almost complete in the order one (there is the proviso of non-void bounded; this can be helped introducing $+\infty$ and $-\infty$).

The completeness of $(\mathbb{R}, |x-y|)$ is in the Bolzano-Cauchy theorem.

Observation. *Each Cauchy sequence is bounded.*

Proof. Consider n_0 such that for all $m, n \geq n_0$, $|x_m - x_n| < 1$. Set $K = \max\{|x_1 - x_n| \,|\, n \leq n_0\}$. Then for every n, $|x_1 - x_n| < K + 1$.

Lemma. *Let $(x_n)_n$ be Cauchy and let a subsequence $(x_{k_n})_n$ converge to x. Then $\lim_n x_n = x$.*

Proof. For an $\varepsilon > 0$ choose an n_1 such that for $n \geq n_1$, $|x_{k_n} - a| < \frac{\varepsilon}{2}$, and an n_2 such that for $m, n \geq n_1$,$|x_n - x_m| < \frac{\varepsilon}{2}$. Then, as $k_n \geq n$, for $n \geq n_0 = \max(n_1, n_2)$, $|x_n - a| \leq |x_n - x_{k_n}| + |x_{k_n} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Corollary. (Bolzano-Cauchy) *Each Cauchy sequence in $\mathbb{R}$ converges. Thus, $\mathbb{R}$ is a complete metric space.*

Example:
Absolutely convergent series.

Suppose we know that $\sum_{n=1}^{\infty} b_n$ converges (for instance, $b_n = Bq^n$ with $0 < q < 1$), and that $|a_n| \leq b_n$ for all n. Then for the partial sums $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ we have

$$|s_m - s_n| = |\sum_{k=n+1}^{m} a_n| \leq \sum_{k=n+1}^{m} |a_n| \leq$$
$$\leq \sum_{k=n+1}^{m} b_n = t_m - t_n$$

$(t_n)_n$ converges, hence it is Cauchy, hence $(s_n)_n$ is Cauchy, and hence $(s_n)_n$ converges.

Thus we know that

a finite sum $\sum_{n=1}^{\infty} a_n$ *exists,*

without having the slightest idea about its value.

Neighborhoods, open sets.

$\Omega(x,\varepsilon) = \{y \mid d(x,y) < \varepsilon\}$

U is a nbhood of $x \equiv \exists\varepsilon, \Omega(x,\varepsilon) \subseteq U$

U is *open* $\equiv U$ is a neighborhood of each of its points.

Fact. *Each* $\Omega(x,\varepsilon)$ *is open.*

(If $y \in \Omega(x,\varepsilon)$, $d(x,y) < \varepsilon$. Choose $\eta > 0$, $\eta < \varepsilon - d(x,y)$. If $z \in \Omega(y,\eta)$ then $d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \eta < \varepsilon$, hence $z \in \Omega(x,\varepsilon)$.)

Image and preimage.

$f[A] = \{f(x) \mid x \in A\}$,

$f^{-1}[B] = \{x \mid f(x) \in B\}$

Facts. $f[A] \subseteq B$ iff $A \subseteq f^{-1}[B]$,

$$f[f^{-1}[B]] \subseteq B, \quad A \subseteq f^{-1}[f[A]].$$

Theorem. *TFAE for* $f : X \to Y$

(1) f *is continuous.*

(2) *for every* $x \in X$ *and every neigborhood* V *of* $f(x)$ *exists* U *neighborhood of* x *such that* $f[U] \subseteq V$*.*

(3) *for every* V *open in* Y*,* $f^{-1}[V]$ *is open in* X*.*

Proof. (1)$\Leftrightarrow$(2): $f[\Omega(x,\delta)] \subseteq \Omega(f(x),\varepsilon)$ says precisely that

$$d(x,y) < \delta \;\Rightarrow\; d(f(x), f(y)) < \varepsilon.$$

(2)$\Rightarrow$ (3): Let V be open and let $x \in f^{-1}[V]$. Then $f(x) \in V$, V is a neighborhood of $f(x)$ and there is a neighborhood U of x such that $f[U] \subseteq V$. Then $x \in U \subseteq f^{-1}[V]$ and $f^{-1}[V]$ is a neighborhood of x.

(3)$\Rightarrow$(2): Let V be a neighborhood of $f(x)$, let $f(x) \in W = \Omega(f(x),\varepsilon) \subseteq V$. W is open, hence $U = f^{-1}[W]$ is open, $x \in U$, and $f[U] = f[f^{-1}[W]] \subseteq W \subseteq V$.

Homeomorphism.

Continuous $f : (X, d) \to (Y, d')$ such that there is a continuous $g : (Y, d') \to (X, d)$ with

$$f \cdot g = \mathrm{id}_Y \quad \text{and} \quad g \cdot f = \mathrm{id}_X.$$

Like isomorphism in algebras and elsewhere.

Example.
$\tan : (-\pi, \pi) \to \mathbb{R}$, $\arctan : \mathbb{R} \to (-\pi, \pi)$

Every open interval (bounded or unbounded) is homeomorphic with $\mathbb{R}$*.*

Equivalent metrics d, d' on X:

$$\mathrm{id} : (X, d) \to (X, d') \text{ is a homeomorphism}$$

Example.
$d(x, y) = |\arctan(x) - \arctan(y)|$
on $\mathbb{R}$ is equivalent with the standard $|x - y|$.

Topological concepts: concepts preserved under homeomorphism;
on a given set: independent on choice of equivalent metric.

Examples. Topological:

continuity

convergence

neighborhood (although $\Omega(x, \varepsilon)$ is not)

open set

closed set

closure

compactness

Not topological:

boundedness

Cauchy property

completeness

(a bounded open interval (a, b) is bounded, $\mathbb{R}$ not. $\mathbb{R}$ is complete, (a, b) is not.)

Strongly equivalent metrics.

Strongly quivalent metrics d, d' on X: exist $\alpha, \beta > 0$ such that

$$\forall x, y, \alpha d(x,y) \leq d'(x,y) \leq \beta d(x,y)$$

Strong equivalence preserves also the boundedness, Cauchy property, or completeness mentioned above.

In particular the metrics on $\mathbb{E}_n$

$$d((x_i)_i, (y_i)_i) = \sqrt{\sum_i (x_i - y_i)^2}$$
$$\lambda((x_i)_i, (y_i)_i) = \sum_i |x_i - y_i|$$
$$\sigma((x_i)_i, (y_i)_i) = \max_i |x_i - y_i|$$

are strongly equivalent.

Similarly the variants of distance in (finite) products:

$$d((x_i)_i,(y_i)_i) = \sqrt{\sum_i d_i(x_i,y_i)^2}$$
$$\lambda((x_i)_i,(y_i)_i) = \sum_i d_i(x_i,y_i)$$
$$\sigma((x_i)_i,(y_i)_i) = \max_i d_i(x_i,y_i)$$

are strongly equivalent:

Obviously we have

$$\max_i d_i(x_i,y_i) \le \sqrt{\sum_i d_i(x_i,y_i)^2}$$
$$\max_i d_i(x_i,y_i) \le \sum_i d_i(x_i,y_i)$$
$$\sqrt{\sum_i d_i(x_i,y_i)^2} \le \sqrt{n}\max_i d_i(x_i,y_i)$$
$$\sum_i d_i(x_i,y_i) \le n\max_i d_i(x_i,y_i)$$

A reformulation of compactness. Accumulation points.

x is an *accumulation point* of M in (X, d) if for every nbhood U of x, $U \cap M$ is infinite.
(Equivalently, if every nbhood U of x contains a $y \neq x$.)

Proposition. *(X, d) is compact iff every infinite $M \subseteq X$ has an accumulation point.*

Proof. I. Let (X, d) be compact, $M \subseteq X$ infinite. Choose a sequence $(x_n)_n$ with all the elements $x_n \in M$ distinct. Let $(x_{k_n})_n$ be a convergent subsequence. Then $x = \lim_n x_{k_n}$ is obviously an accumulation point of M.

II. Conversely, let $(x_n)_n$ be in X. If $\{x_n \mid n = 1, 2, \dots\}$ is finite, there is a constant subsequence. Else $M = \{x_n \mid n = 1, 2, \dots\}$ has an accumulation point x, and every $M \cap \Omega(x, \frac{1}{n})$ is infinite. Choose $x_{k_1} \in M \cap \Omega(x, 1)$. If we have chosen x_{k_1} with $k_1 < k_2 < \cdots < k_n$ such that $x_{k_r} \in M \cap \Omega(x, \frac{1}{r})$ choose $x_{k_{n+1}} \in M \cap \Omega(x, \frac{1}{n+1})$ with $k_{n+1} > k_n$ (we have still infinitely many candidates). Then $\lim_n x_{k_n} = x$.

Details.

Text:

Chapter I, Section 2

Chapter II, Section 3

Chapter III, Section 2

Chapter XIII, Sections 2, 3, 4, 6 and 7