Repetition.
Multivariable Riemann integral.

Until the theorem of existence of integral for continuous functions everything went smoothly.

An *n-dimensional compact interval* (a *brick*) in $\mathbb{E}_n$ is

$$J = \langle a_1, b_1 \rangle \times \cdots \times \langle a_n, b_n \rangle.$$

A *partition* of J is a sequence $P = (P^1, \ldots, P^n)$ of partitions

$P^j: \; a_j = t_{j0} < t_{j1} < \cdots < t_{j,n_j-1} < t_{j,n_j} = b_j,$

Bricks of the partition P are the

$$\langle t_{1,i_1}, t_{1,i_1+1} \rangle \times \cdots \times \langle t_{n,i_n}, t_{n,i_n+1} \rangle$$

and

$$\mathcal{B}(P)$$

is the set of all the bricks of P.

It is an *almost disjoint* decomposition of J.

Mesh of a partition P. For $J = \langle r_1, s_1\rangle \times \cdots \times \langle r_n, s_n\rangle$:

$$\mathsf{diam}(J) = \max_i (s_i - r_i);$$

and the *mesh* of P is:

$$\mu(P) = \max\{\mathsf{diam}(B) \mid B \in \mathcal{B}(P)\}.$$

Refinement. A partition $Q = (Q^1, \ldots, Q^n)$ *refines* a partition $P = (P^1, \ldots, P^n)$ if every Q^j refines P^j.

Again, trivially, for any two partitions P, Q of an n-dimensional compact interval J there is a common refinement.

For $f : J \to \mathbb{R}$ on J set

$$m(f, B) = \inf\{f(\mathbf{x}) \,|\, \mathbf{x} \in B\} \quad \text{and}$$
$$M(f, B) = \sup\{f(\mathbf{x}) \,|\, \mathbf{x} \in B\},$$

and for a partition P of J and a bounded function $f : J \to \mathbb{R}$ set

$$s(f, P) = \sum\{m(f, B) \cdot \mathsf{vol}(B) \,|\, B \in \mathcal{B}(P)\},$$
$$S(f, P) = \sum\{M(f, B) \cdot \mathsf{vol}(B) \,|\, B \in \mathcal{B}(P)\}.$$

Easily, using a common reinement, we obtain

Proposition. *Let P, Q be arbitrary partitions of J. Then we have*

$$s(f, P) \leq S(f, Q).$$

Hence we can define the lower Riemann integral

$$\underline{\int}_J f(\mathbf{x})\mathrm{d}\mathbf{x} = \sup\{s(f,P) \mid P \text{ a partition}\};$$

and the upper one

$$\overline{\int}_J f(\mathbf{x})\mathrm{d}\mathbf{x} = \inf\{S(f,P) \mid P \text{ a partition}\},$$

and if they are equal we call the common value the Riemann integral

$$\int_J f(\mathbf{x})\mathrm{d}\mathbf{x} \quad \text{or simply} \quad \int_J f$$

Another notation:

$$\int_J f(x_1, \ldots, x_n)\mathrm{d}x_1, \ldots x_n$$

or

$$\int_J f(x_1, \ldots, x_n)\mathrm{d}x_1\mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

Proposition. *Riemann integral $\int_J f(\mathbf{x})d\mathbf{x}$ exists if and only if for every $\varepsilon > 0$ there is a partition P such that*

$$S(f, P) - s(f, P) < \varepsilon.$$

From this (and uniform continuity) we obtain, quite like in one variable

Theorem. *For every continuous function $f : J \to \mathbb{R}$ on an n-dimensional compact interval the Riemann integral $\int_J f$ exists.*

What we DO NOT have is a counterpart of the Fundamental Theorem of Calculus, in particular its consequence that

for any primitive function G of f we can compute the Riemann integral in one variable as

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

.

The means to compute the Riemann integral in several variables will be now provided by Fubini Theorem.

Theorem. (Fubini) *Consider the product* $J = J' \times J'' \subseteq \mathbb{E}_{m+n}$ *of intervals* $J' \subseteq \mathbb{E}_m$, $J'' \subseteq \mathbb{E}_n$. *Let* $\int_J f(\mathbf{x},\mathbf{y})d\mathbf{xy}$ *exist and let for every* $\mathbf{x} \in J'$ *(resp. every* $\mathbf{y} \in J''$*) the integral* $\int_{J''} f(\mathbf{x},\mathbf{y})d\mathbf{y}$ *(resp.* $\int_{J'} f(\mathbf{x},\mathbf{y})d\mathbf{x}$*) exist. Then*

$$\int_J f(\mathbf{x},\mathbf{y})d\mathbf{xy} = \int_{J'}(\int_{J''} f(\mathbf{x},\mathbf{y})d\mathbf{y})d\mathbf{x} = \\ = \int_{J''}(\int_{J'} f(\mathbf{x},\mathbf{y})d\mathbf{x})d\mathbf{y}.$$

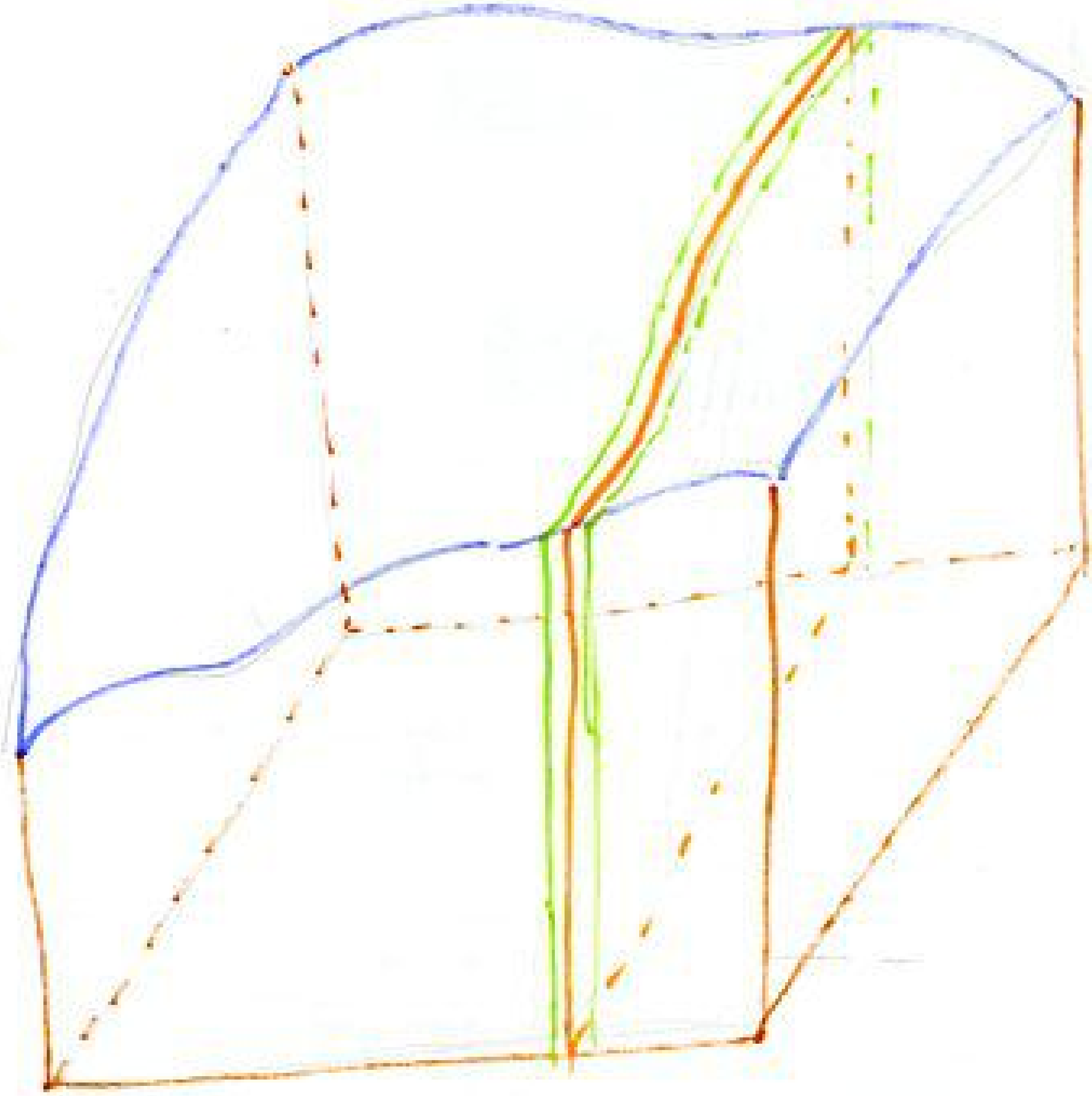
Thus in two variables

$$\int_J f = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\mathrm{d}y\right)\mathrm{d}x,$$

in three variables

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{a_3}^{b_3} f(x_1,x_2,x_3)\mathrm{d}x_3\right)\mathrm{d}x_2\right)dx_1$$

etc.



Before the proof: No surprise, it is quite like

$$\sum_{i\le m, j\le n} x_{ij} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right)$$

and this is in fact what is being done, with lower and upper sums, in the proof.

Proof of Theorem. Set

$$F(\mathbf{x}) = \int_{J''} f(\mathbf{x}, \mathbf{y}) \mathrm{d}\mathbf{y}.$$

We will prove that $\int_{J'} F$ exists and that

$$\int_J f = \int_{J'} F.$$

Choose a partition P of J such that

$$\int f - \varepsilon \le s(f, P) \le S(f, P) \le \int f + \varepsilon.$$

This partition P is constituted of a partition P' of J' and a partition P'' of J''. We have

$$\mathcal{B}(P) = \{B' \times B'' \mid B' \in \mathcal{B}(P'), B'' \in \mathcal{B}(P'')\},$$

and each brick of P appears as precisely one $B' \times B''$.
Then

$$F(\mathbf{x}) \leq \sum_{B'' \in \mathcal{B}(P'')} \max_{\mathbf{y} \in B''} f(\mathbf{x}, \mathbf{y}) \cdot \mathsf{vol} B''$$

and hence

$$\begin{aligned} S(F, P') &\leq \sum_{B' \in \mathcal{B}(P')} \max_{\mathbf{x} \in B'} \Big(\sum_{B'' \in \mathcal{B}(P'')} \max_{\mathbf{y} \in B''} f(\mathbf{x}, \mathbf{y}) \cdot \mathsf{vol}(B'')\Big) \cdot \mathsf{vol}(B') \leq \\ &\leq \sum_{B' \in \mathcal{B}(P')} \sum_{B'' \in \mathcal{B}(P'')} \max_{(\mathbf{x}, \mathbf{y}) \in B' \times B''} f(\mathbf{x}, \mathbf{y}) \cdot \mathsf{vol}(B'') \cdot \mathsf{vol}(B') \leq \\ &\leq \sum_{B' \times B'' \in \mathcal{B}(P)} \max_{\mathbf{z} \in B' \times B''} f(\mathbf{z}) \cdot \mathsf{vol}(B' \times B'') = S(f, P), \end{aligned}$$

and similarly

$$s(f, P) \leq s(F, P').$$

Hence we have

$$\int_j f - \varepsilon \leq s(F, P') \leq \int_{J'} F \leq S(F, P) \leq \int_J f + \varepsilon$$

and therefore $\int_{J'} F$ is equal to $\int_J f$.

Example: Volume of a ball. On the interval $J = \langle -r, r\rangle \times \langle -r, r\rangle$ consider

$$f(x,y) = \begin{cases} \sqrt{r^2 - x^2 - y^2} \text{ if } r^2 - x^2 - y^2 \geq 0, \\ 0 \text{ otherwise} \end{cases}$$

The volume of the ball will be $2\int_J f$, hence

$$2\int_{-r}^{r}\left(\int_{-r}^{r} f(x,y)\mathrm{d}y\right)\mathrm{d}x = 2\int_{-r}^{r}\left(\int_{-u}^{u}\sqrt{r^2 - x^2 - y^2}\mathrm{d}y\right)\mathrm{d}x$$

where $u = \sqrt{r^2 - x^2}$. First compute the

$$\int_{-u}^{u}\sqrt{r^2 - x^2 - y^2}\mathrm{d}y = \int_{-u}^{u}\sqrt{u^2 - y^2}\mathrm{d}y = u\int_{-u}^{u}\sqrt{1 - (\frac{y}{u})^2}\mathrm{d}y.$$

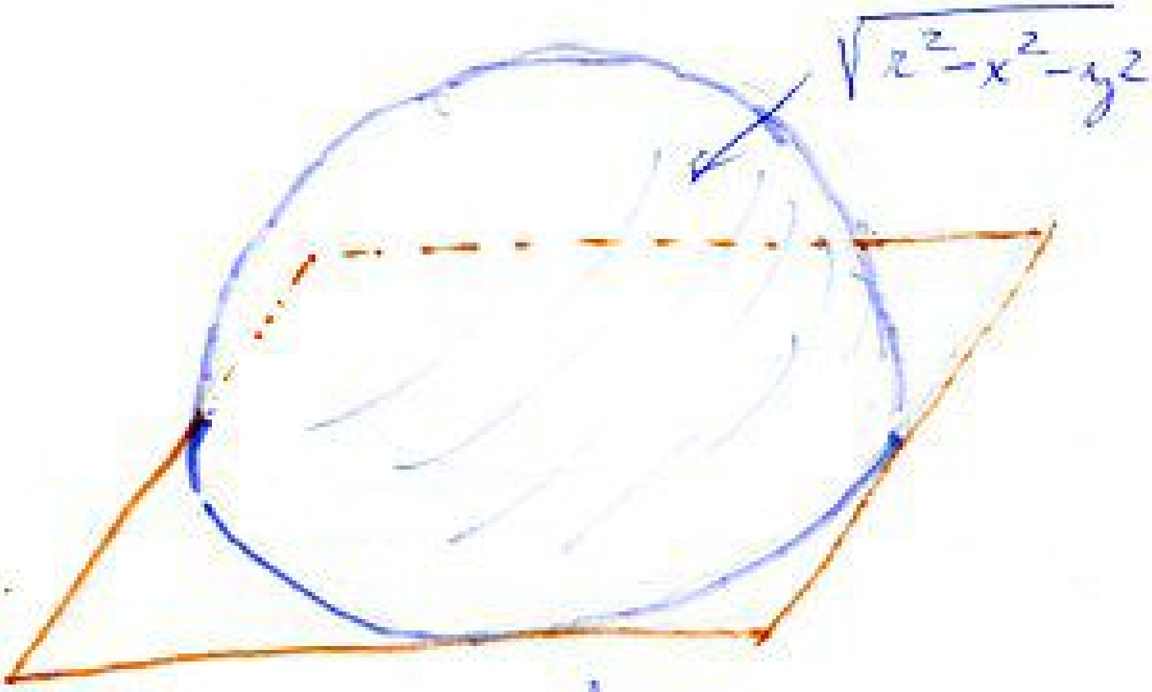
Substitution $\frac{y}{u} = \sin t$ yields $\mathrm{d}y = u\cos t\mathrm{d}t$ and

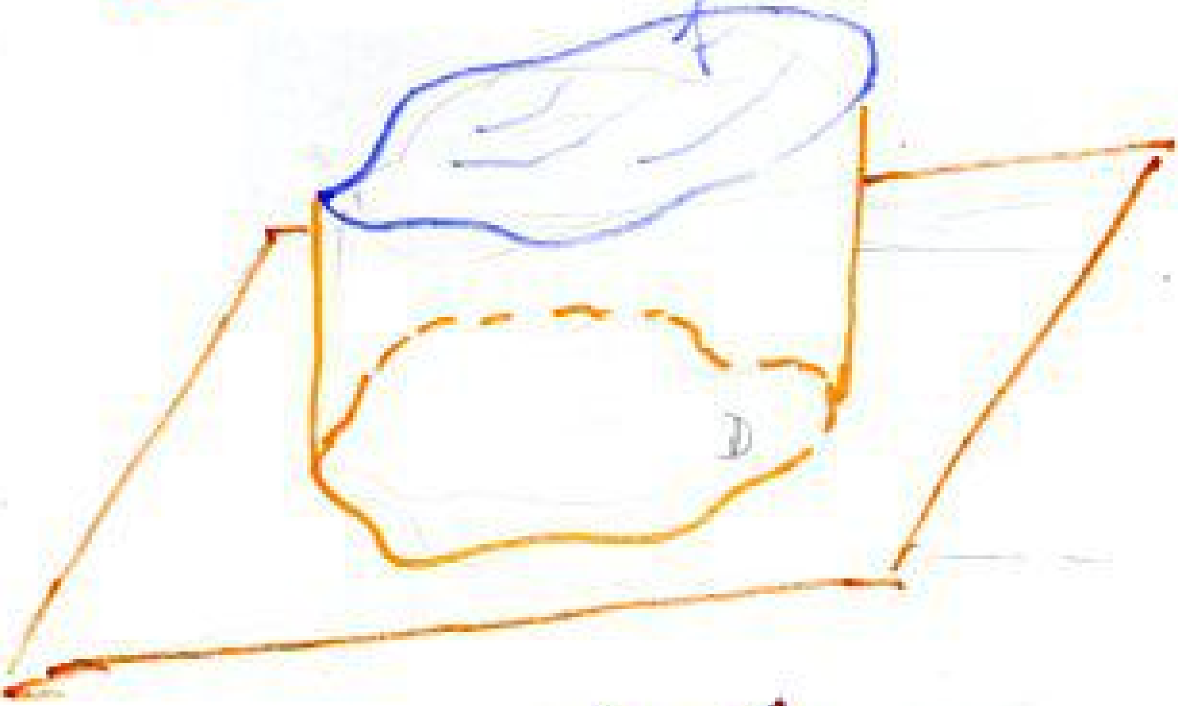
$$u\int_{-u}^{u}\sqrt{1 - (\frac{y}{u})^2}\mathrm{d}y = u^2\int_{-\pi}^{\pi}\sqrt{1 - \sin^2 t}\cos t\mathrm{d}t =$$

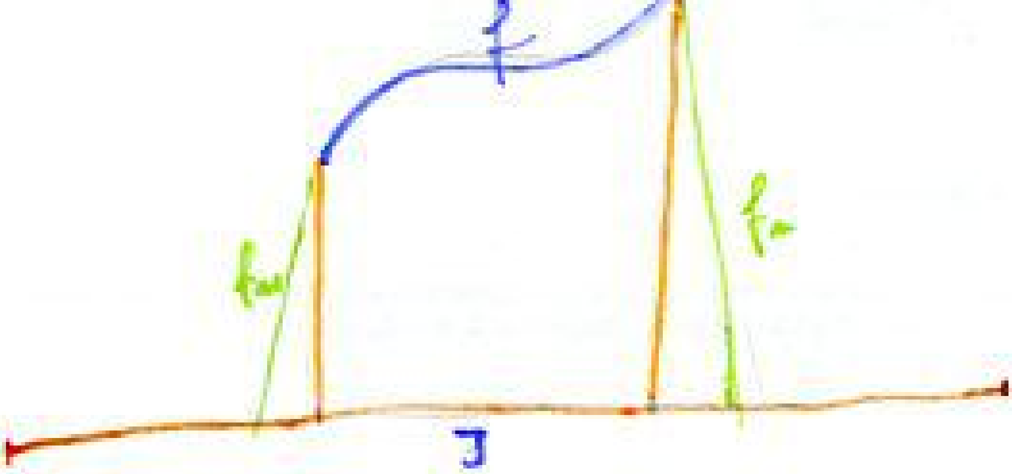
$$= u^2\int_{-\pi}^{\pi}\cos^2 t\mathrm{d}t = u^2\pi = (r^2 - x^2)\pi$$

using the primitive function $\frac{\cos 2t+1}{2}$ of $\cos^2 t$. Thus, the volume of the ball is

$$2\pi\int_{-2}^{r}(r^2 - x^2)\mathrm{d}x = 2\pi(r^3 - \frac{1}{3}r^3) = \frac{4}{3}\pi r^3.$$


$\sqrt{r^2-x^2-y^2}$


f
D


f
f_n
f_n
J

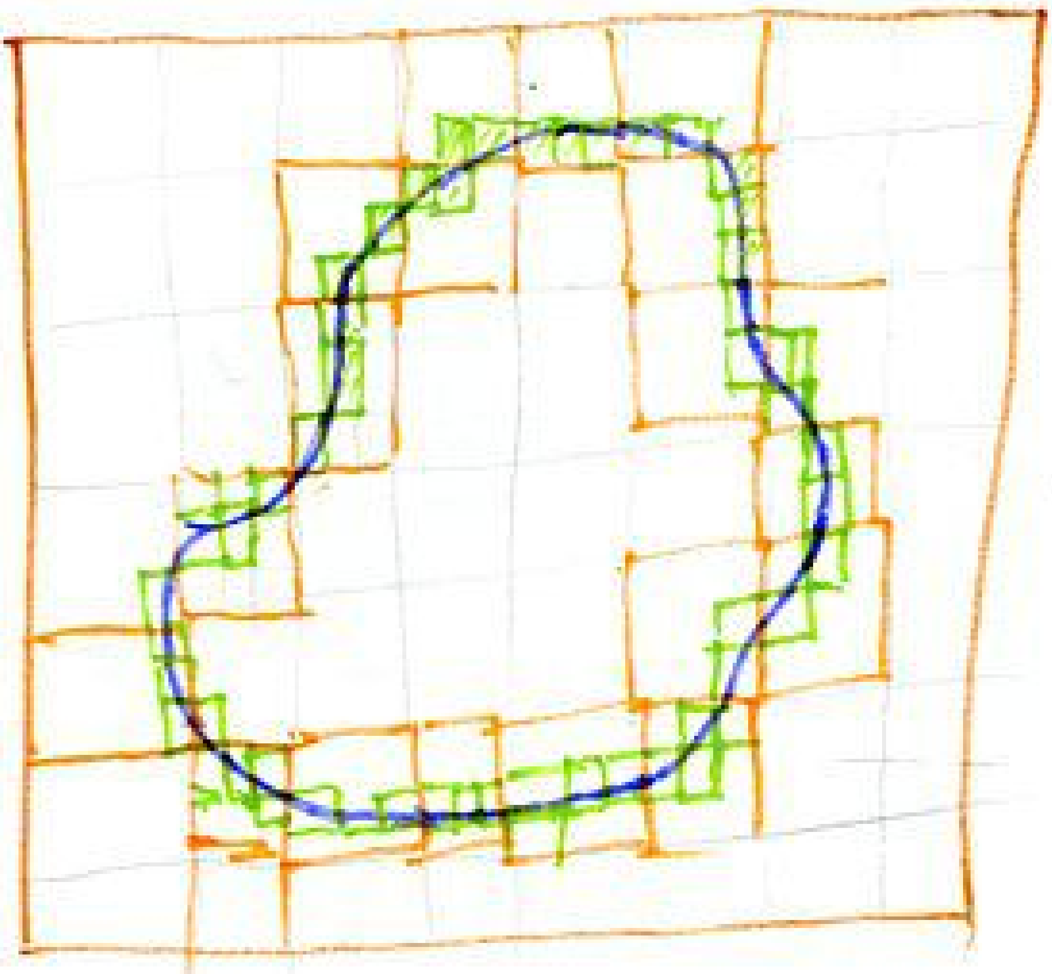
The example also reveals another problem with the integral in n variables. Unlike real functions in one variable where an interval was a sufficiently typical domain, a function in n variables is not typically defined on a brick. What we do with such a function defined, say, on a compact (that is, closed bounded D) is that we

- first embed the D into a brick J,
- and then extend f by values 0 on $J \setminus D$.

But there is still a problem. Does thus extended function have a Riemann integral? In the example we were lucky: $\sqrt{r^2 - x^2 - y^2}$ thus extended from the ball $\{(x,y) \mid x^2+y^2 \leq r^2\}$ to the square was continuous.

In general, one typicaly uses Fubini theorem and in the individual variables one encounters Riemann integral of functions in one variable that are discontinuous in finitely many points, which is OK (EXPLAIN).

This is not quite correct: Fubini Theorem, as we have it, assumes the existence of the whole integral $\int_J f(\mathbf{x}, \mathbf{y}) \mathrm{d}\mathbf{xy}$, not only of all the (say) $F(\mathbf{x}) = \int_{J''} f(\mathbf{x}, \mathbf{y}) \mathrm{d}\mathbf{y}$ plus the existence of $\int_{J'} F(\mathbf{x}) \mathrm{d}\mathbf{x}$. Most usually, however, the existence of $\int_J f(\mathbf{x}, \mathbf{y}) \mathrm{d}\mathbf{xy}$ follows from the fact that althought the number of the discontinuity points is infinite, they are in the border of D and the border od D has visibly volume 0.



First remark on Lebesgue integral.

Riemann integral can be extended so that, e.g., it is correct to compute

$$\int \lim f_n = \lim \int f_n$$

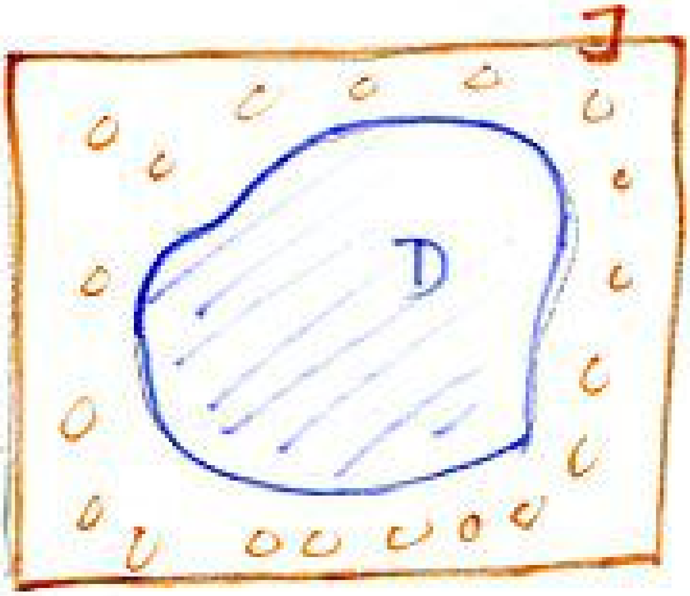
just under the assumption that (e.g.) $|f_n(\mathbf{x})| \leq K$ (K a constant).

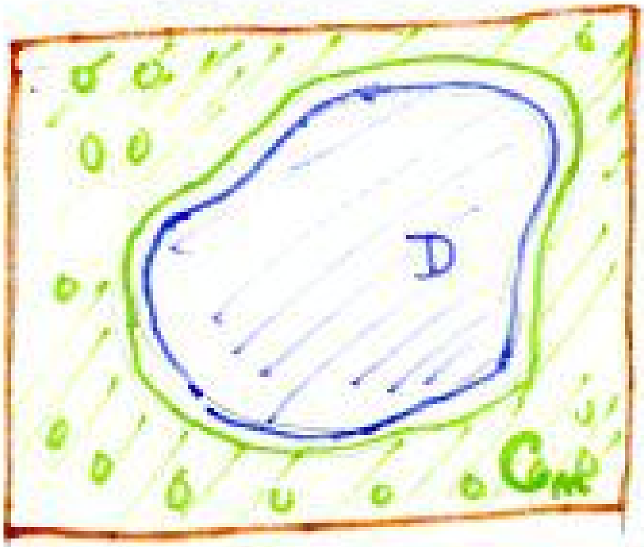
For a closed D subset of a compact n-dimensional interval J and a function f continuous on D, as follows. Define

$$J_n = \{x \,|\, d(x, D) \geq \frac{1}{n}\}.$$

Then g_n defined on J_n as 0, and as f on D, is continuous and we can extend it to equally bounded continuous f_n on J (Tietze Theorem). Then

$\lim f_n$ is f on D and 0 otherwise.





$C_m = \{x \mid d(x,D) \geq \frac{1}{m}\}$

f_m ... $f_m|D = f$

continuous $f_m|C_m = \text{const}_0$

Note. There is no extension of an integral in which

$$\int \lim f_n = \lim \int f_n$$

would hold unconditionally. Consider on (say) $\langle -1, 1 \rangle$

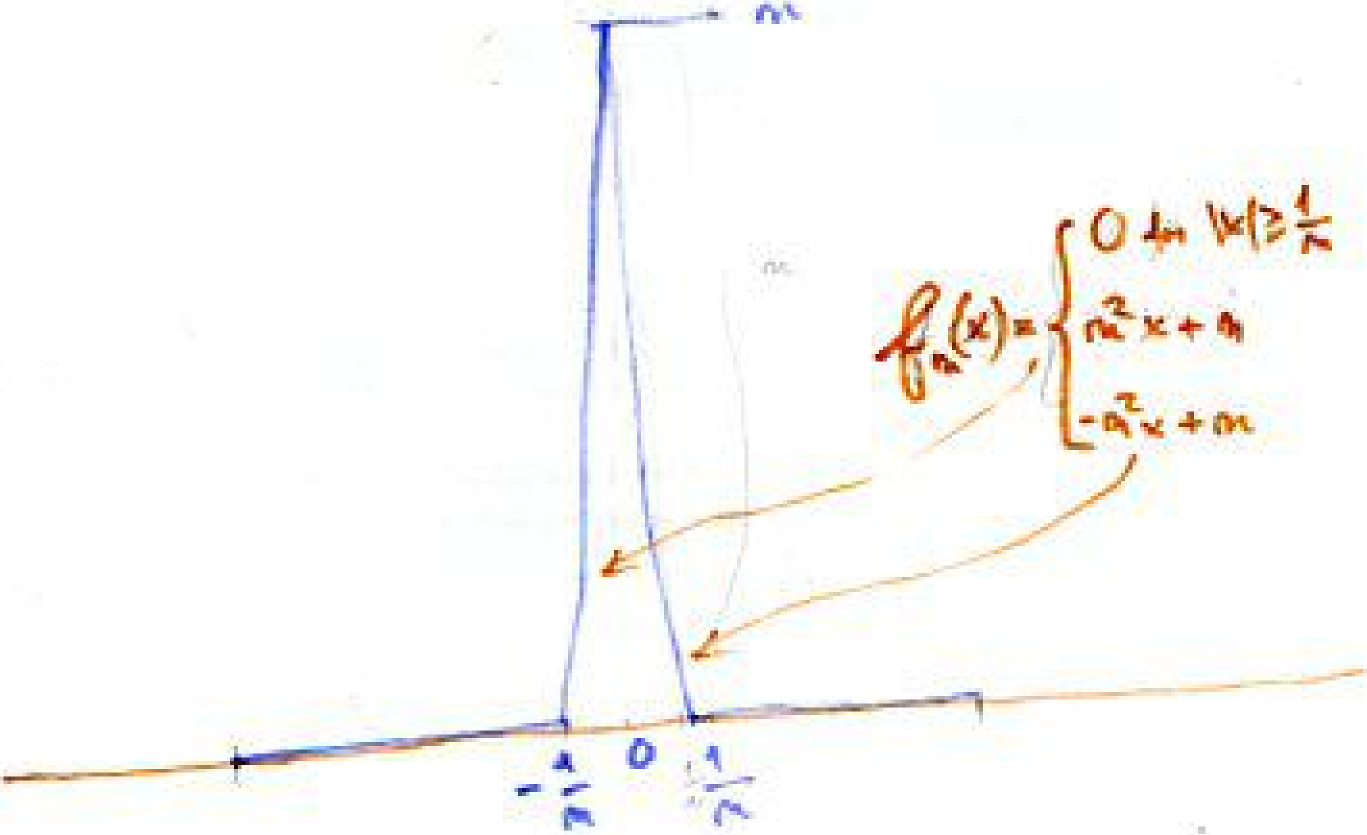
$$f_n(x) = \begin{cases} 0 \text{ for } |x| \geq \frac{1}{n} \\ n^2x + n \text{ for } -\frac{1}{n} \leq x \leq 0 \\ -n^2x + n \text{ for } 0 \leq x \leq \frac{1}{n} \end{cases}$$

and

$$g_n(x) = \begin{cases} 0 \text{ for } x \neq 0 \\ n \text{ for } x = 0 \end{cases}$$

Then

$$\int f_n = 1, \quad \int g_n = 0 \quad \text{and} \quad \lim f_n = \lim g_n.$$



$$g_n(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ n & \text{for } x = 0 \end{cases}$$

$$\int f_n(x)\,dx = 1 \qquad \int g_n(x)\,dx = 0$$

! $\lim f_n = \lim g_n$!

Details.
Text: Chapter XVI, Section 4