Repetition.

Uniform continuity. $f : (X,d) \to (Y,d')$ is *uniformly continuous* if

$$\forall\varepsilon\exists\delta \text{ s.t. } d(x,y) < \delta \Rightarrow d'(f(x),f(y)) < \varepsilon.$$

More precisely, quantifying x and y:

$$\forall\varepsilon\exists\delta \text{ s.t. } \forall x \forall y\, d(x,y) < \delta \Rightarrow d'(f(x),f(y)) < \varepsilon.$$

The position of $\forall x$ is essential.

Plain continuity requires

$$\forall x \forall\varepsilon\exists\delta \text{ s.t. } \forall y \ \cdots$$

Uniform continuity is stronger, but we have

Theorem. *If (X,d) is compact then every continuous $f : (X,d) \to (Y,d')$ is uniformly continuous.*

In particular this holds for continuous real functions on compact intervals.

Volumes (areas) (for $A \subseteq \mathbb{E}_n$)

Properties:

- $A \subseteq B \quad \Rightarrow \quad \mathsf{vol}(A) \leq \mathsf{vol}(B)$
- A, B *disjoint* $\Rightarrow \mathsf{vol}(A \cup B) = \mathsf{vol}(A) + \mathsf{vol}(B)$
- vol *is preserved under isometry*
- *in* $\mathbb{E}_n$ *:*
  $\mathsf{vol}(\prod_i \langle a_i, b_i \rangle) = (b_1 - a_1) \cdot \cdots \cdot (b_n - a_n)$

Fact. *Generally*

$\mathsf{vol}(A \cup B) = \mathsf{vol}(A) + \mathsf{vol}(B) - \mathsf{vol}(A \cap B)$.

Volume of a facet of a brick is zero, hence the volume of a system of bricks intersecting just in facets is the sum of their volumes.

(We speak of almost disjoint unions.)

Riemann integral in one variable:

A *partition* of $\langle a, b\rangle$

$P:\ a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$

Refinements.

Mesh of P, $\mu(P) = \max_j(t_j - t_{j-1})$.

Lower and upper sums

$$s(f, P) = \sum_{j=1}^{n} m_j(t_j - t_{j-1}) \quad \text{resp.}$$

$$S(f, P) = \sum_{j=1}^{n} M_j(t_j - t_{j-1})$$

with

$$m_j = \inf\{f(x) \,|\, t_{j-1} \le x \le t_j\},$$
$$M_j = \sup\{f(x) \,|\, t_{j-1} \le x \le t_j\}.$$

Lower resp. *upper Riemann integral* of a function f

$$\underline{\int_a^b} f(x)\mathrm{d}x = \sup\{s(f,P) \mid P \text{ a partition}\} \quad \text{and}$$

$$\overline{\int_a^b} f(x)\mathrm{d}x = \inf\{S(f,P) \mid P \text{ a partition}\}.$$

If they are equal,

$$\int_a^b f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x$$

is the <u>Riemann integral of f over $\langle a,b\rangle$</u>.

If $f(x) \geq 0$ on $\langle a,b\rangle$ then $\int_a^b f(x)dx$ *is the area (volume) of*

$$\{(x,y) \mid a \leq x \leq b, x \leq y \leq f(x)\}$$

Proposition. *Riemann integral* $\int_a^b f(x)dx$ *exists if and only if for every* $\varepsilon > 0$ *there is a partition* P *such that*

$$S(f,P) - s(f,P) < \varepsilon.$$

Theorem. *For every continuous function* $f : \langle a,b\rangle \to \mathbb{R}$ *Riemann integral* $\int_a^b f$ *exists.*

Theorem. (Integral Mean Value Thm) *Let* $f : \langle a,b\rangle \to \mathbb{R}$ *be continuous. Then there exists a* $c \in \langle a,b\rangle$ *such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

Theorem. (Fund. Thm of Calculus) *Let* $f : \langle a, b\rangle \to \mathbb{R}$ *be continuous. For* $x \in \langle a, b\rangle$ *set*

$$F(x) = \int_a^x f(t)dt.$$

Then $F'(x) = f(x)$.

Corollary. *Let* $f : \langle a, b\rangle \to \mathbb{R}$ *be continuous. Then it has a primitive function on* (a, b) *continuous on* $\langle a, b\rangle$. *If* G *is any primitive function of* f *on* (a, b) *continuous on* $\langle a, b\rangle$ *then*

$$\int_a^b f(t)dt = G(b) - G(a).$$

.

Corollary. (Integral mean value thm:)

$$F(b)-F(a) = \int_a^b f = f(c)(b-a) = F'(c)(b-a)$$

Multivariable Riemann integral.

In $\mathbb{E}_n$, a *compact interval* (an *n-dimensional compact interval*) is

$$J = \langle a_1, b_1\rangle \times \cdots \times \langle a_n, b_n\rangle$$

(indeed it is compact); briefly, an *interval*, or a *brick*.

A *partition* of J is a sequence $P = (P^1, \ldots, P^n)$ of partitions

$$P^j : a_j = t_{j0} < t_{j1} < \cdots < t_{j,n_j-1} < t_{j,n_j} = b_j,$$

The intervals

$$\langle t_{1,i_1}, t_{1,i_1+1}\rangle \times \cdots \times \langle t_{n,i_n}, t_{n,i_n+1}\rangle$$

will be called the *bricks of* P, and

$$\mathcal{B}(P)$$

is the set of all the bricks of P.

It is an *almost disjoint* decomposition of J.

That is, distinct bricks in $\mathcal{B}(P)$ obviously meet in a subset of a facet, hence in a set of volume 0. Hence we have

Observation.
$\mathsf{vol}(J) = \sum\{\mathsf{vol}(B) \mid B \in \mathcal{B}(J)\}$.

Mesh of a partition.
diameter of $J = \langle r_1, s_1\rangle \times \cdots \times \langle r_n, s_n\rangle$:

$$\mathsf{diam}(J) = \max_i (s_i - r_i);$$

the *mesh* of a partition P:

$$\mu(P) = \max\{\mathsf{diam}(B) \mid B \in \mathcal{B}(P)\}.$$

Refinement. A partition $Q = (Q^1, \ldots, Q^n)$ *refines* a partition $P = (P^1, \ldots, P^n)$ if every Q^j refines P^j.

Observation. *A refinement Q of a partition P induces partitions*

$$Q_B \quad \textit{of the bricks} \quad B \in \mathcal{B}(P)$$

and we have an almost disjoint union

$$\mathcal{B}(Q) = \bigcup \{\mathcal{B}(Q_B) \mid B \in \mathcal{B}(P)\}.$$

Observation. *For any two partitions P, Q of an n-dimensional compact interval J there is a common refinement.*

$f : J \to \mathbb{R}$ is defined on an n-dimensional compact interval J, bounded, and $B \subseteq J$ is an n-dimensional compact subinterval of J. Set

$$m(f, B) = \inf\{f(\mathbf{x}) \mid \mathbf{x} \in B\} \quad \text{and}$$
$$M(f, B) = \sup\{f(\mathbf{x}) \mid \mathbf{x} \in B\}.$$

Fact. *$m(f, B) \leq M(f, B)$ and if $C \subseteq B$ then*

$$m(f, C) \geq m(f, B) \text{ and } M(f, C) \leq M(f, B).$$

For a partition P of an interval J and a bounded function $f : J \to \mathbb{R}$ set

$$s_J(f, P) = \sum\{m(f, B) \cdot \mathsf{vol}(B) \mid B \in \mathcal{B}(P)\},$$
$$S_J(f, P) = \sum\{M(f, B) \cdot \mathsf{vol}(B) \mid B \in \mathcal{B}(P)\}.$$

A general observation:

$f : X \to \mathbb{R}$ bounded,

$X = \bigcup X_i$, $X_i = \bigcup X_{ij}$

finite (almost) disjoint unions

$$M_i = \sup\{f(x) \,|\, x \in X_i\},$$
$$M_{ij} = \sup\{f(x) \,|\, x \in X_{ij}\}$$

Trivially: $M_{ij} \leq M_i$

(M_i is an upper bound of $\{f(x) \,|\, x \in X_{ij}\}$).

Hence

$$\sum_i M_i \mathsf{vol}(X_i) = \sum_i M_i \sum_j \mathsf{vol}(X_{ij}) =$$
$$= \sum_{ij} M_i \mathsf{vol}(X_{ij}) \geq \sum_{ij} M_{ij} \mathsf{vol}(X_{ij})$$

Similarly for infima.

Proposition. *Let a partition Q refine P. Then*

$$s(f,Q) \geq s(f,P) \quad \textit{and} \quad S(f,Q) \leq S(f,P).$$

Proof: Apply the observation above for $\{X_i \,|\, i\} = \mathcal{B}(P)$, $\{X_{ij} \,|\, j\} = \mathcal{B}(Q_B)$, and of course $\{X_{ij} \,|\, ij\} = \mathcal{B}(Q)$.

Proposition. *Let P, Q be arbitrary partitions of J. Then we have*

$$s(f,P) \leq S(f,Q).$$

Proof. Since trivially $s(f,P) \leq S(f,P)$, if we consider a common refinement R of P, Q we obtain

$$s(f,P) \leq s(f,R) \leq S(f,R) \leq S(f,Q).$$

Hence: The set $\{s(f,P)\,|\,P \text{ a partition}\}$ is bounded from above and we can define the *lower Riemann integral* of f over J as

$$\underline{\int_J} f(\mathbf{x})\mathrm{d}\mathbf{x} = \sup\{s(f,P)\,|\,P \text{ a partition}\};$$

similarly,

$$\overline{\int_J} f(\mathbf{x})\mathrm{d}\mathbf{x} = \inf\{S(f,P)\,|\,P \text{ a partition}\}.$$

If they are equal we call the common value the *Riemann integral of f over J*, denoted

$$\int_J f(\mathbf{x})\mathrm{d}\mathbf{x} \quad \text{or simply} \quad \int_J f$$

Another notation

$$\int_J f(x_1, \ldots, x_n) \mathrm{d}x_1, \ldots x_n$$

or

$$\int_J f(x_1, \ldots, x_n) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

It makes more sense than meets the eye.

Obviously we have the simple estimate

$$\inf\{f(\mathbf{x}) \,|\, \mathbf{x} \in J\} \cdot \mathsf{vol}(J) \le \underline{\int}_J f \le$$

$$\le \overline{\int}_J f \le \sup\{f(\mathbf{x}) \,|\, \mathbf{x} \in J\} \cdot \mathsf{vol}(J).$$

Proposition. *Riemann integral* $\int_J f(\mathbf{x}) d\mathbf{x}$ *exists if and only if for every* $\varepsilon > 0$ *there is a partition* P *such that*

$$S_J(f, P) - s_J(f, P) < \varepsilon.$$

Note that it is quite straightforward: the inequality yields

$$S_J(f, P) < \varepsilon + s_J(f, P)$$

and from this

$$\overline{\int} \leq S_J(f, P) < \varepsilon + s_J(f, P) \leq \varepsilon + \underline{\int} \leq \varepsilon + \overline{\int}$$

and ε is arbitrarily small.

Theorem. *For every continuous function $f : J \to \mathbb{R}$ on an n-dimensional compact interval the Riemann integral $\int_J f$ exists.*

Proof. We will use the distance σ in $\mathbb{E}_n$ defined by

$$\sigma(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|.$$

Since f is uniformly continuous we can choose for $\varepsilon > 0$ a $\delta > 0$ such that

$$\sigma(\mathbf{x}, \mathbf{y}) < \delta \quad \Rightarrow \quad |f(\mathbf{x}) - f(\mathbf{y})| < \frac{\varepsilon}{\mathsf{vol}(J)}.$$

Recall the mesh $\mu(P)$. If $\mu(P) < \delta$ then $\mathsf{diam}(B) < \delta$ for all $B \in \mathcal{B}(P)$ and hence

$$M(f,B) - m(f,B) = \sup\{f(\mathbf{x}) \,|\, \mathbf{x} \in B\} - \inf\{f(\mathbf{x}) \,|\, \mathbf{x} \in B\} \leq$$
$$\leq \sup\{|f(\mathbf{x}) - f(\mathbf{y})| \,|\, \mathbf{x}, \mathbf{y} \in B\} < \frac{\varepsilon}{\mathsf{vol}(J)}$$

so that

$$S(f,P) - s(f,P) =$$
$$= \sum\{(M(f,B) - m(f,B)) \cdot \mathsf{vol}(B) \,|\, B \in \mathcal{B}(P)\} \leq$$
$$\leq \frac{\varepsilon}{\mathsf{vol}(J)} \sum\{\mathsf{vol}(B) \,|\, B \in \mathcal{B}(P)\} = \frac{\varepsilon}{\mathsf{vol} J}\mathsf{vol}(J) = \varepsilon.$$

Details.

Text: Chapter XVI, Sections 1,2,3

Chapter XIII, 2.3