Repetition.

General Implicit Functions Theorem.

Theorem. *Let* $F_i(\mathbf{x}, y_1, \ldots, y_m)$, $i = 1, \ldots, m$, *be functions of* $n+m$ *variables with continuous partial derivatives up to an order* $k \geq 1$. *Let*

$$\mathbf{F}(\mathbf{x}^0, \mathbf{y}^0) = \mathbf{o}$$

and let

$$\frac{\mathsf{D}(\mathbf{F})}{\mathsf{D}(\mathbf{y})}(\mathbf{x}^0, \mathbf{y}^0) \neq 0.$$

Then there exist $\delta > 0$ *and* $\Delta > 0$ *such that for every*

$$\mathbf{x} \in (x_1^0 - \delta, x_1^0 + \delta) \times \cdots \times (x_n^0 - \delta, x_n^0 + \delta)$$

there exists precisely one

$$\mathbf{y} \in (y_1^0 - \Delta, y_1^0 + \Delta) \times \cdots \times (y_m^0 - \Delta, x_m^0 + \Delta)$$

such that

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0.$$

(That is,

$$\begin{aligned} &F_1(\mathbf{x}, y_1, \ldots, y_n) = 0, \\ &\cdots \quad \cdots \quad \cdots \\ &F_n(\mathbf{x}, y_1, \ldots, y_n) = 0. \quad) \end{aligned}$$

Furthermore, if we write this $\mathbf{y}$ *as a vector function* $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, *then the functions* f_i *have continuous partial derivatives up to the order* k.

Jacobi determinant.

For

$$\mathbf{F}(\mathbf{x},\mathbf{y}) = (F_1(\mathbf{x}, y_1, \ldots, y_m), \ldots, F_m(\mathbf{x}, y_1, \ldots, y_m)).$$

and $\mathbf{y} = (y_1, \ldots, y_m)$:

Jacobi determinant (*Jacobian*)

$$\frac{\mathsf{D}(\mathbf{F})}{\mathsf{D}(\mathbf{y})} = \det \left(\frac{\partial F_i}{\partial y_j} \right)_{i,j=1,\ldots,m}$$

Recall:

(The absolute value) of the determinant

$$\begin{vmatrix} a_{11}, a_{12}, \ldots, a_{1n} \\ \ldots, \ldots, \ldots \\ a_{n1}, a_{n2}, \ldots, a_{nn} \end{vmatrix} \neq 0.$$

is the volume of the parallelepiped determined by the vectors $(a_{11}, a_{12}, \ldots, a_{1n})$,..., $(a_{11}, a_{12}, \ldots, a_{1n})$.

Hence, Jacobian expreses local changes of volume under transformation $\mathbf{F}(\mathbf{x}, -)$.

Application: Extremes under constraints.

Theorem. *Let $f, g_1, \ldots, g_k$ be real functions defined in an open set $D \subseteq \mathbb{E}_n$, and let them have continuous partial derivatives. Suppose that the rank of the matrix*

$$M = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1}, & \ldots, & \dfrac{\partial g_1}{\partial x_n} \\ \ldots, & \ldots, & \ldots \\ \dfrac{\partial g_k}{\partial x_1}, & \ldots, & \dfrac{\partial g_k}{\partial x_n} \end{pmatrix}$$

is the largest possible, that is k, everywhere in D.

If the function f achieves at a point $\mathbf{a} = (a_1, \ldots, a_n)$ a local extreme subject to the constraints

$$g_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, k$$

then there exist numbers $\lambda_1, \ldots, \lambda_k$ such that for each $i = 1, \ldots, n$ we have

$$\frac{\partial f(\mathbf{a})}{\partial x_i} + \sum_{j=1}^{k} \lambda_j \cdot \frac{\partial g_j(\mathbf{a})}{\partial x_i} = 0.$$

Principle of proof (how the IFT is applied):

Let, say, the leftmost square submatrice of M be regular. Thus we have

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_1}, & \ldots, & \frac{\partial g_1}{\partial x_k} \\ \ldots, & \ldots, & \cdots \\ \frac{\partial g_k}{\partial x_1}, & \ldots, & \frac{\partial g_k}{\partial x_k} \end{vmatrix} \neq 0. \qquad (*)$$

Think of the constraints ($k < n$)

$$g_1(x_1, \ldots, x_n) = 0$$
$$\cdots \quad \cdots \quad \cdots$$
$$g_k(x_1, \ldots, x_n) = 0$$

as of an Implicit Functions task

$$g_i(\overbrace{x_1, \ldots, x_k}^{\mathbf{y}}, x_{k+1}, \ldots, x_n) = 0, \; i = 1, \ldots, k$$

Then $(*)$ amounts to $\frac{\mathsf{D}(\mathbf{g})}{\mathsf{D}(\mathbf{y})} \neq 0$ and we have for $i \leq k$, $x_i = \phi_i(x_{k+1}, ..., x_n)$ with $\widetilde{\mathbf{x}} = x_{k+1}, ..., x_n$ moving freely in

$$F(x_{k+1}, \ldots, x_n) = f(\phi_1(\widetilde{\mathbf{x}}), \ldots, \phi_k(\widetilde{\mathbf{x}}), \widetilde{\mathbf{x}}).$$

The rest is Chain Rule and linear algebra (and the primitive search for local extremes by zero partial derivatives).

Regular maps.

For open $U \subseteq \mathbb{E}_n$ and

$$f_i : U \to \mathbb{R}, \quad i = 1, \ldots, n,$$

the resulting mapping

$$\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{E}_n$$

is called *regular* if

$$\frac{\mathsf{D}(\mathbf{f})}{\mathsf{D}(\mathbf{x})}(\mathbf{x}) \neq 0$$

for all $\mathbf{x} \in U$.

Proposition. *If $\mathbf{f} : U \to \mathbb{E}_n$ is regular then the image $\mathbf{f}[V]$ of every open $V \subseteq U$ is open.*

Proposition. *Let $\mathbf{f} : U \to \mathbb{E}_n$ be a regular mapping. Then for each $\mathbf{x} \in U$ there exists an open neighborhood V such that the restriction $\mathbf{f}|V$ is one-to-one. Moreover, the mapping $\mathbf{g} : f[V] \to \mathbb{E}_n$ inverse to $\mathbf{f}|V$ is regular.*

Corollary. *A one-to-one regular mapping $\mathbf{f} : U \to \mathbb{E}_n$ has a regular inverse $\mathbf{g} : \mathbf{f}[U] \to \mathbb{E}_n$.*

Volumes (areas).

$A \subseteq \mathbb{E}_n$ (in particular, $\mathbb{E}_2$)

Properties:

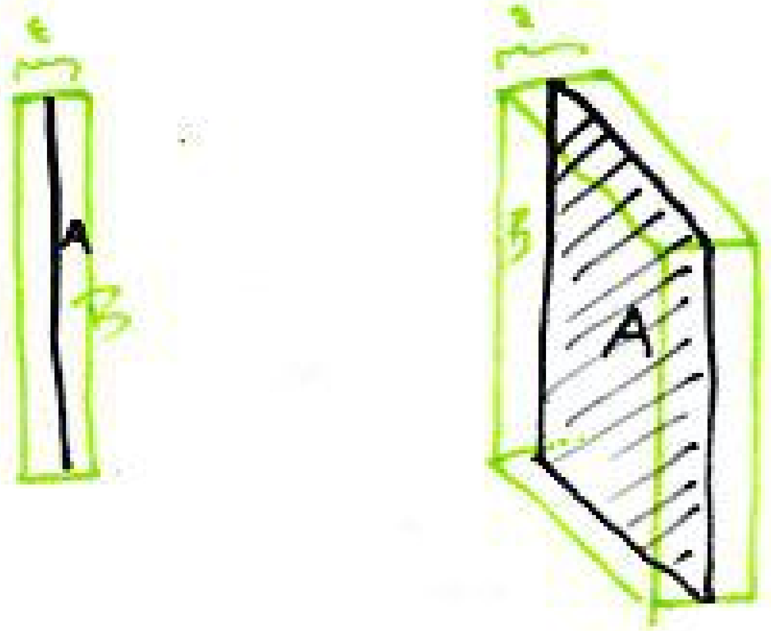
- $A \subseteq B \quad \Rightarrow \quad \mathsf{vol}(A) \leq \mathsf{vol}(B)$
- A, B disjoint $\Rightarrow \mathsf{vol}(A \cup B) = \mathsf{vol}(A) + \mathsf{vol}(B)$
- vol is preserved under isometry.

- In $\mathbb{E}_2$:
  $\mathsf{vol}(\langle a_1, b_1\rangle \times \langle a_2, b_2\rangle) = (b_1 - a_1)(b_2 - a_2)$
- In $\mathbb{E}_n$:
  $\mathsf{vol}(\prod_i \langle a_i, b_i\rangle) = (b_1 - a_1) \cdot \cdots \cdot (b_n - a_n)$

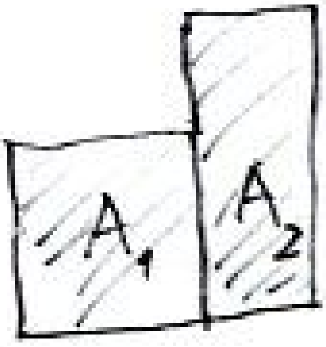
Fact. Generally

$$\mathsf{vol}(A \cup B) = \mathsf{vol}(A) + \mathsf{vol}(B) - \mathsf{vol}(a \cap B).$$

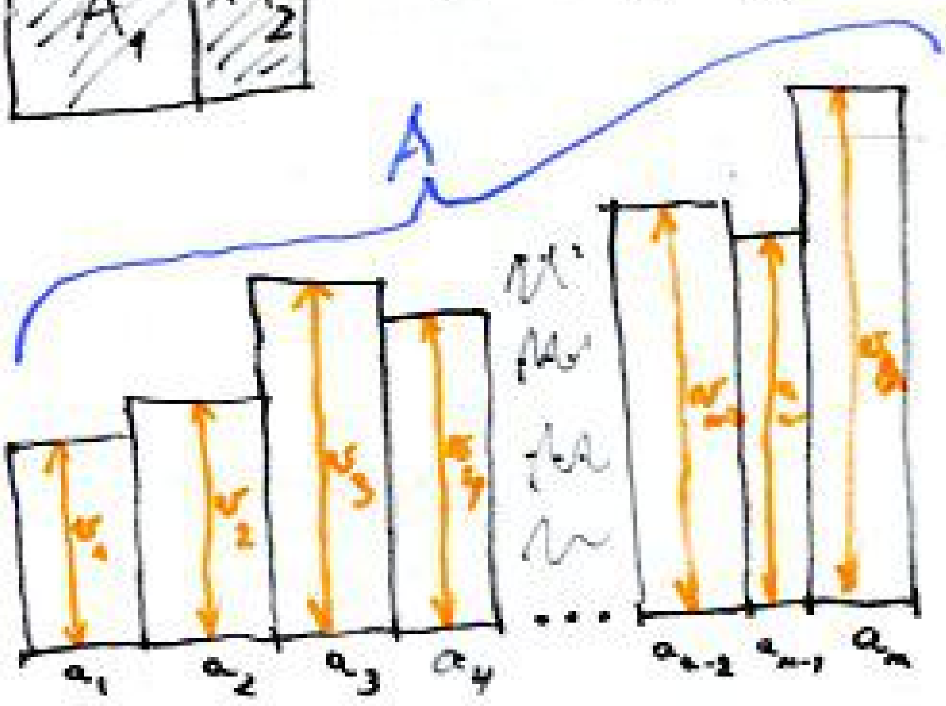
(Combine the disjoint unions $A \cup B = A \cup (B \smallsetminus A)$ and $B = (B \smallsetminus A) \cup (A \cap B)$.)



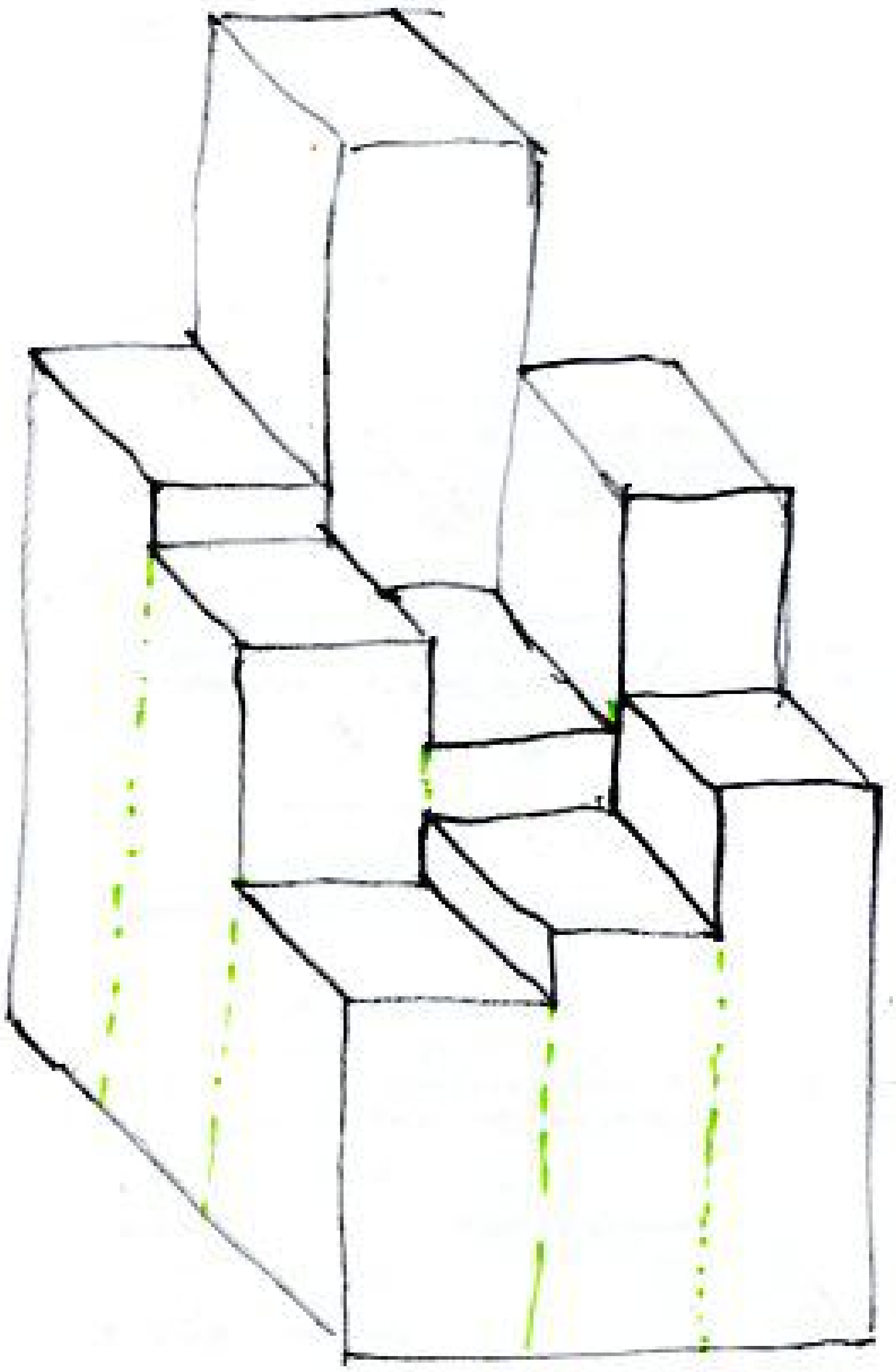
$$\mathrm{vol}(A) \leq \mathrm{vol}(B) = \begin{cases} a \cdot \varepsilon \\ a_1 \cdot a_2 \cdot \varepsilon \end{cases}$$



$$\mathrm{vol}(A_1 \cup A_2) = \mathrm{vol}\, A_1 + \mathrm{vol}\, A_2$$



$$\mathrm{vol}(A) = \sum_{i=1}^{n} a_i v_i$$



Uniform continuity. $f : (X, d) \to (Y, d')$ is *uniformly continuous* if

$\forall \varepsilon \exists \delta$ s.t. $d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$.

Compare with plain continuity: If we consequentially quantify the points x, y, plain continuity is defined by

$$\forall x \forall \varepsilon \exists \delta \text{ s.t. } \forall y \; d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon.$$

while the new uniform continuity has

$$\forall \varepsilon \exists \delta \text{ s.t. } \forall x \forall y \; d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon.$$

The position of the quantifier $\forall x$ is essential !

Example. $f = (x \mapsto x^2) : \mathbb{R} \to \mathbb{R}$ is continuous but not uniformly continuous.

We have $|f(x) - f(y)| = |x + y| \cdot |x - y|$; thus, to have $|f(x) - f(y)| < \varepsilon$ in the vicinity of $x = 100$ we need the δ 100 times smaller than if we are in the vicinity of $x = 1$.

But we have

Theorem. *If* (X, d) *is compact then every continuous* $f : (X, d) \to (Y, d')$ *is uniformly continuous.*

In particular *this holds for continuous real functions on compact intervals.*

Proof. Let $f : (X, d) \to (Y, d')$ not be uniformly continuous. Then there is an $\varepsilon > 0$ such that for every n there are x_n, y_n with

$$d(x_n, y_n) < \frac{1}{n} \qquad (*)$$

and

$$d'(f(x_n), f(y_n)) \geq \varepsilon. \qquad (**)$$

Choose a convergent subsequence $(x_{k_n})_n$ of $(x_n)_n$. Set $a = \lim_n x_{k_n}$. Then by $(*)$ also $a = \lim_n y_{k_n}$. By $(**)$ we cannot have both $f(a) = \lim_n f(x_{k_n})$ and $f(a) = \lim_n f(y_{k_n})$, and hence f is not continuous.

Riemann integral in one variable, recapitulation.

A *partition* of $\langle a, b\rangle$: it is a sequence

$$P: \; a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Refinement:

$$P': \; a = t'_0 < t'_1 < \cdots < t'_{n-1} < t_m = b$$

s. t. $\{t_j \,|\, j = 1, \ldots, n-1\} \subseteq \{t'_j \,|\, j = 1, \ldots, m-1\}$.

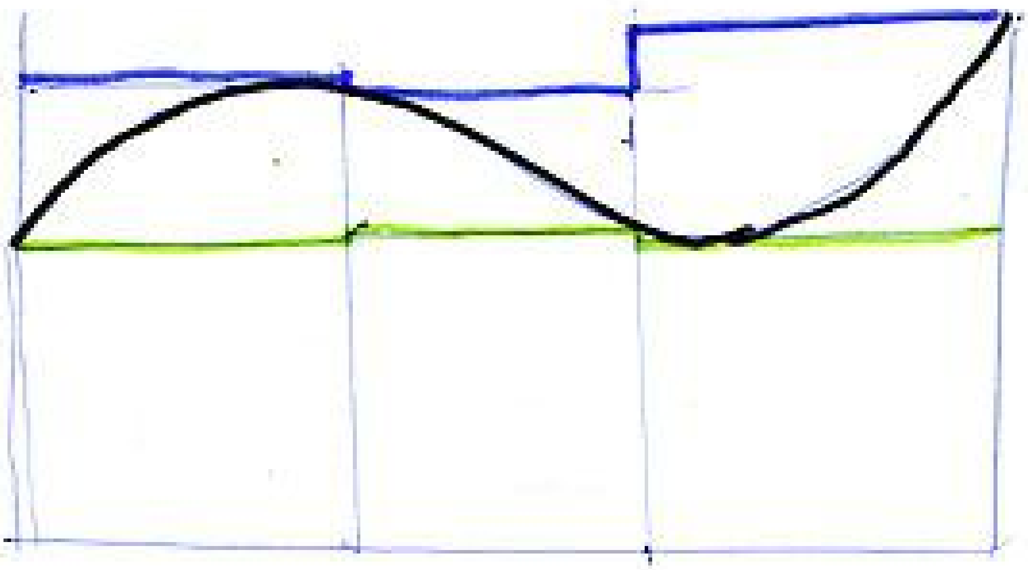
Mesh of P, $\mu(P) = \max_j(t_j - t_{j-1})$.

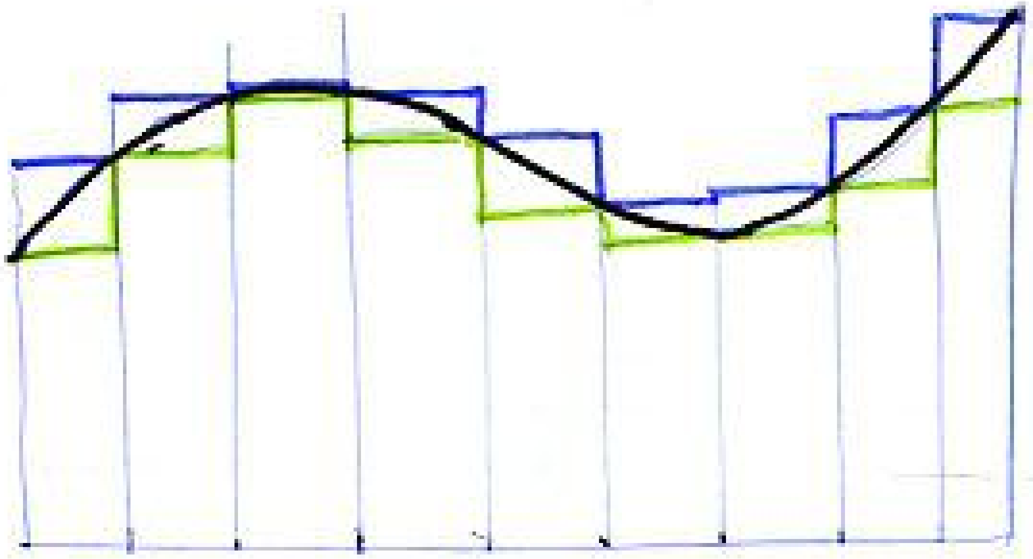
For bounded $f: J = \langle a, b\rangle \to \mathbb{R}$ and P as above: Define lower and upper sums

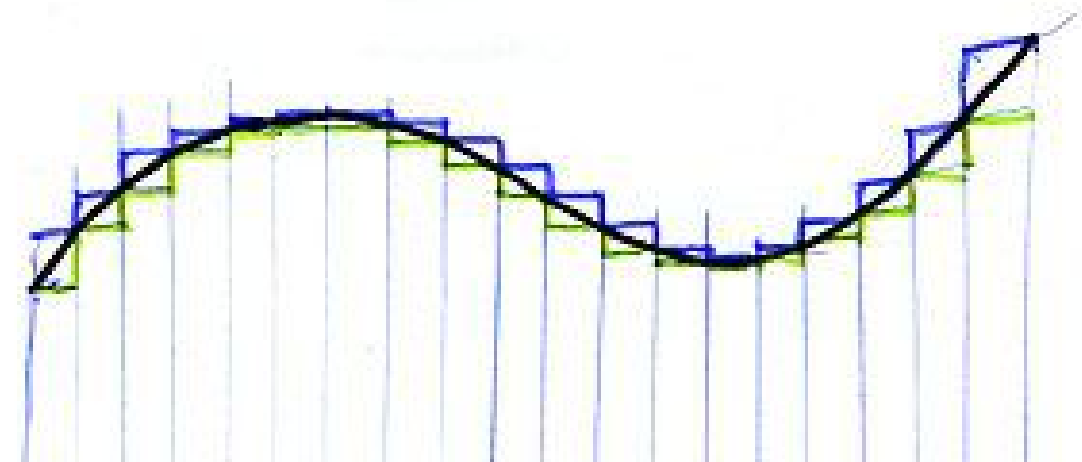
$$s(f, P) = \sum_{j=1}^{n} m_j(t_j - t_{j-1}) \quad \text{resp.}$$
$$S(f, P) = \sum_{j=1}^{n} M_j(t_j - t_{j-1})$$

with

$$m_j = \inf\{f(x) \,|\, t_{j-1} \le x \le t_j\}, M_j = \sup\{f(x) \,|\, t_{j-1} \le x \le t_j\}.$$







Easy facts. 1. *If P' refines P then*
$$s(f,P) \le s(f,P') \quad \textit{and} \quad S(f,P) \ge S(f,P')$$

2. *For any two P_1, P_2,*
$$s(f,P_1) \le S(f,P_2).$$

The integral.

$$\underline{\int_a^b} f(x)\mathrm{d}x = \sup\{s(f,P) \mid P \text{ a partition}\} \text{ and}$$
$$\overline{\int_a^b} f(x)\mathrm{d}x = \inf\{S(f,P) \mid P \text{ a partition}\}.$$

The first is called the *lower Riemann integral* of f over $\langle a,b\rangle$, the second is the *upper Riemann integral* of f.

If $\underline{\int}_a^b f(x)\mathrm{d}x = \overline{\int}_a^b f(x)\mathrm{d}x$ then the common value is denoted by

$$\int_a^b f(x)\mathrm{d}x$$

and called *Riemann integral* of f over $\langle a, b\rangle$.

Proposition. *Riemann integral* $\int_a^b f(x)dx$ *exists if and only if for every* $\varepsilon > 0$ *there is a partition* P *such that*

$$S(f, P) - s(f, P) < \varepsilon.$$

Proof. I. Let $\int_a^b f(x)\mathrm{d}x$ exist and let $\varepsilon > 0$. Then there are partitions P_1 and P_2 such that

$$S(f, P_1) < \int_a^b f(x)\mathrm{d}x + \frac{\varepsilon}{2} \quad \text{and} \quad s(f, P_2) > \int_a^b f(x)\mathrm{d}x + \frac{\varepsilon}{2}.$$

Then we have for the common refinement P of P_1, P_2,

$$S(f, P) - s(f, P) < \int_a^b f(x)\mathrm{d}x + \frac{\varepsilon}{2} - \int_a^b f(x)\mathrm{d}x + \frac{\varepsilon}{2} = \varepsilon.$$

II. Let the statement hold. Choose an $\varepsilon > 0$ such that $S(f,P) - s(f,P) < \varepsilon$. Then

$$\overline{\int_a^b} f(x)\mathrm{d}x \leq S(f,P) < s(f,P) + \varepsilon \leq \underline{\int_a^b} f(x)\mathrm{d}x + \varepsilon,$$

and since ε was arbitrary we conclude that $\overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x$.

Theorem. *For every continuous function $f : \langle a,b\rangle \to \mathbb{R}$ Riemann integral $\int_a^b f$ exists.*

Proof. For $\varepsilon > 0$ choose $\delta > 0$ s.t.

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

If $\mu(P) < \delta$ we have $t_j - t_{j-1} < \delta$ for all j, and hence

$$M_j - m_j = \sup\{f(x) \,|\, t_{j-1} \leq x \leq t_j\} - \inf\{f(x) \,|\, t_{j-1} \leq x \leq t_j\} \leq$$
$$\leq \sup\{|f(x) - f(y)| \,|\, t_{j-1} \leq x, y \leq t_j\} \leq \frac{\varepsilon}{b-a}$$

so that

$$S(f,P) - s(f,P) = \sum (M_j - m_j)(t_j - t_{j-1}) \leq$$
$$\leq \frac{\varepsilon}{b-a} \sum (t_j - t_{j-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Theorem. (Integral Mean Value Thm) *Let* $f : \langle a, b\rangle \to \mathbb{R}$ *be continuous. Then there exists a* $c \in \langle a, b\rangle$ *such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof. Set $m = \min\{f(x) \,|\, a \leq x \leq b\}$ and $M = \max\{f(x) \,|\, a \leq x \leq b\}$. Obviously

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a).$$

Hence there is a K with $m \leq K \leq M$ such that $\int_a^b f(x)\mathrm{d}x = K(b-a)$. Since f is continuous there is a $c \in \langle a, b\rangle$ such that $K = f(c)$.

Observation. *For any $a < b < c$,*

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

Theorem. (Fund. Thm of Calculus) *Let $f : \langle a, b\rangle \to \mathbb{R}$ be continuous. For $x \in \langle a, b\rangle$ set*

$$F(x) = \int_a^x f(t)dt.$$

Then $F'(x) = f(x)$.

Proof. For $h \neq 0$ we have

$$\frac{1}{h}(F(x+h) - f(x)) = \frac{1}{h}(\int_a^{x+h} f - \int_a^x f) =$$

$$= \frac{1}{h}\int_x^{x+h} f = \frac{1}{h} f(x+\theta h)h = f(x+\theta h)$$

where $0 < \theta < 1$ and as f is continuous, $\lim_{h\to 0} \frac{1}{h}(F(x+h) - f(x)) = \lim_{h\to 0} f(x+\theta h) = f(x)$.

Corollary. *Let* $f : \langle a, b\rangle \to \mathbb{R}$ *be continuous. Then it has a primitive function on* (a, b) *continuous on* $\langle a, b\rangle$*. If* G *is any primitive function of* f *on* (a, b) *continuous on* $\langle a, b\rangle$ *then*

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

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Details.

Text: Chapter XI, Sections 1, 2, 3 and 4.