Repetition.

Compact space: every $(x_n)_n$ in (X, d) has a convergent subsequence.

From previous semester: $\langle a, b\rangle$ is compact, hence the term "compact interval"

A closed subspace of a compact space is compact.

Bounded X. Products.

A subspace of $\mathbb{E}_n$ is compact iff it is closed and bounded.

The image of a compact subset under a continuous map is compact.

Corollary. *A continuous real function on a compact space attains a maximum and a minimum.*

(a compact $M \subseteq \mathbb{R}$ has the largest and least elements: it is bounded and closed)

A compact subset of an arbitrary metric space is closed.

Corollary. *If X is compact and $f : X \to Y$ continuous, then for every $A \subseteq X$ the image $f[A]$ is closed in Y.*

What is peculiar, discuss.

Cauchy sequences. Convergent sequences are Cauchy.

Complete space. $\mathbb{R}$ is complete.

A product of complete spaces is complete. $\mathbb{E}_n$ is complete.

A subspace of $\mathbb{E}_n$ is complete iff it is closed.

Every compact space is complete.

Implicit Functions Theorems.

The Task: Given continuous real functions $F_i(x_1,\ldots,x_m,y_1,\ldots,y_n)$, $i=1,\ldots,n$ of $m+n$ variables. Does the system of equations

$$F_1(x_1,\ldots,x_m,y_1,\ldots,y_n)=0,$$
$$\ldots \quad \ldots \quad \ldots$$
$$F_n(x_1,\ldots,x_m,y_1,\ldots,y_n)=0$$

determine in some sense functions

$$f_i \equiv y_i(x_1,\ldots,x_m),\ \ i=1,\ldots,n,$$

how, where, and what are their properties?

Illustrate the problems on $F(x,y)=x^2+y^2-1$, that is, on the equation

$$x^2+y^2=1$$

(MA5pic 1)

A few observations:

– for some x_0 like the $x_0 < -1$ in the square to the left, there is no solution of $F(x_0, y) = 0$, not to speak of $y(x)$ in the vicinity;

– even if a solution in the neighborhood of an x_0 exists, we cannot correctly think of a function in this neighborhood of x_0. What one needs is a “window” around a solution (x_0, y_0) delimiting not only a neigborhood of x_0 but also a neighborhood of y_0;

– and there is also the case like that of the $x_0 = 1$ where there are plenty of solutions in the vicinity, but no window allowing for unique y's, even one-sided.

In the case of a function $F(x, y)$ this is all that can happen. We have

Theorem. *Let $F(x, y)$ be a real function defined in a nbhood of (x_0, y_0). Let F have continuous partial derivatives up to the order $k \geq 1$ and let*

$$F(x_0, y_0) = 0 \quad \textit{and} \quad \left|\frac{\partial F(x_0, y_0)}{\partial y}\right| \neq 0.$$

Then there are $\delta > 0$, $\Delta > 0$ s. t. for each $x \in (x_0 - \delta_1, x_0 + \delta_1)$, there is precisely one $y \in (y_0 - \Delta_1, y_0 + \Delta_1)$ s. t.

$$F(x, y) = 0.$$

Furthermore, if we write $y = f(x)$ for this unique y, then this $f : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ has continuous derivatives up to the order k.

Let, say,

$$\frac{\partial F(x_0, y_0)}{\partial y} > 0.$$

$\frac{\partial F}{\partial y}$ are continuous, $A(\delta) = \{x \,|\, |x - x_0| \leq \delta\}$ is closed and bounded, hence compact and there are $a > 0$, K, $\delta_1 > 0$ and $\Delta > 0$ such that for all $(x, y) \in (x_0 - \delta_1, x_0 + \delta_1) \times \langle y_0 - \Delta, y_0 + \Delta\rangle$ we have

$$\frac{\partial F(x, y)}{\partial y} \geq a \quad \text{and} \quad \left|\frac{\partial F(x, y)}{\partial x}\right| \leq K. \tag{$*$}$$

The function f: Fix an $x \in U(\delta_1) = (x_0 - \delta_1, x_0 + \delta_1)$ and define a function in y, $y \in (y_0 - \Delta, y_0 + \Delta)$ by

$$\varphi_x(y) = F(x, y).$$

Then $\varphi'_x(y) = \frac{\partial F(x,y)}{\partial y} > 0$ and hence *all* $\varphi_x(y)$ *are increasing in* y, *and*

$$\varphi_{x_0}(y_0 - \Delta) < \varphi_{x_0}(y_0) = 0 < \varphi_{x_0}(y_0 + \Delta).$$

F is continuous, and hence there is a δ, $0 < \delta \leq \delta_1$, such that

$$\forall x \in U(\delta), \quad \varphi_x(y_0 - \Delta) < 0 < \varphi_x(y_0 + \Delta).$$

φ_x is increasing and hence one-to-one. Thus there is precisely one $y \in (y_0 - \Delta, y_0 + \Delta)$ such that $\varphi_x(y) = 0$ – that is, $F(x, y) = 0$. Denote this y by $f(x)$.

(MA5pic 2)

Properties of f: (Note that so far we do not know even whether f is continuous.)

By Lagrange theorem

$$0 = F(t+h, f(t+h)) - F(t, f(t)) =$$
$$= F(t+h, f(t) + (f(t+h) - f(t))) - F(t, f(t)) =$$
$$= \frac{\partial F(t+\theta h, f(t) + \theta(f(t+h) - f(t)))}{\partial x} h$$
$$+ \frac{\partial F(t+\theta h, f(t) + \theta(f(t+h) - f(t)))}{\partial y} (f(t+h) - f(t))$$

hence

$$f(t+h) - f(t) = -h \cdot \frac{\dfrac{\partial F(t+\theta h, f(t) + \theta(f(t+h) - f(t)))}{\partial x}}{\dfrac{\partial F(t+\theta h f(t) + \theta(f(t+h) - f(t)))}{\partial y}} \qquad (*)$$

for some θ between 0 and 1.Thus,

$$|f(t+h) - f(t)| \leq |h| \cdot \left| \frac{K}{a} \right|$$

Hence f is continuous, and from $(*)$ further

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = -\frac{\dfrac{\partial F(t, f(t))}{\partial x}}{\dfrac{\partial F(t, f(t))}{\partial y}}$$

We have, hence,

$$f'(t) = -\frac{\dfrac{\partial F(t, f(t))}{\partial x}}{\dfrac{\partial F(t, f(t))}{\partial y}}$$

and from this formula we can now take derivatives of higher order as long as the existence of derivatives of the partial derivatives on the right hand side allow.

Note. Hence we can take derivatives of f as long as we can take partial derivatives of F. But beware: there have to be at least the first derivatives (note the $k \geq 1$ in the formulation of Theorem). The $\frac{\partial F}{\partial y}$ was needed already for the existence of f.

There was just one variable in f ($m = 1$ in the problem setting just to avoid complicated notation). The same reasoning yields

Theorem. *Let $F(\mathbf{x}, y)$ be a function of $m+1$ variables defined in a neighbourhood of a point $(\mathbf{x}^0, y_0)$. Let F have continuous partial derivatives up to the order $k \geq 1$ and let*

$$F(\mathbf{x}^0, y_0) = 0 \quad \textit{and} \quad \left| \frac{\partial F(\mathbf{x}^0, y_0)}{\partial y} \right| \neq 0.$$

Then there exist $\delta > 0$ and $\Delta > 0$ such that for every $\mathbf{x}$ with $||\mathbf{x} - \mathbf{x}^0|| < \delta$ there exists precisely one y with $|y - y_0| < \Delta$ such that

$$F(\mathbf{x}, y) = 0.$$

Furthermore, if we write $y = f(\mathbf{x})$ for this unique solution y, then the function

$$f : (x_1^0 - \delta, x_1^0 + \delta) \times \cdots \times (x_n^0 - \delta, x_n^0 + \delta) \to \mathbb{R}$$

has continuous partial derivatives up to the order k.

Two equations

Consider a pair of equations

$$F_1(\mathbf{x}, y_1, y_2) = 0,$$
$$F_2(\mathbf{x}, y_1, y_2) = 0$$

and try to find a solution $y_i = f_i(\mathbf{x})$, $i = 1, 2$, in a neighborhood of a point $(\mathbf{x}^0, y_1^0, y_2^0)$. Apply the Theorem about one equation. Think of the second equation as an equation for y_2; in a neighborhood of $(\mathbf{x}^0, y_1^0, y_2^0)$ we obtain y_2 as a function $\psi(\mathbf{x}, y_1)$. Substituting this into the first equation to obtain

$$G(\mathbf{x}, y_1) = F_1(\mathbf{x}, y_1, \psi(\mathbf{x}, y_1));$$

and a solution $y_1 = f_1(\mathbf{x})$ in a neighborhood of $(\mathbf{x}^0, y_1^0)$ can be substituted into ψ to obtain $y_2 = f_2(\mathbf{x}) = \psi(\mathbf{x}, f_1(\mathbf{x}))$.

What we have assumed:

– continuous partial derivatives of F_i.

– Then, to obtain ψ we needed to have

$$\frac{\partial F_2}{\partial y_2}(\mathbf{x}^0, y_1^0, y_2^0) \neq 0. \qquad (*)$$

– Finally, we need (use the Chain Rule)

$$\frac{\partial G}{\partial y_1}(\mathbf{x}^0, x^0) = \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial y_2}\frac{\partial \psi}{\partial y_1} \neq 0. \qquad (**)$$

Use the formula we already have

$$\frac{\partial \psi}{\partial y_1} = -\left(\frac{\partial F_1}{\partial y_2}\right)^{-1}\frac{\partial F_2}{\partial y_1}$$

and transform $(**)$ to

$$\left(\frac{\partial F_1}{\partial y_2}\right)^{-1}\left(\frac{\partial F_1}{\partial y_1}\frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2}\frac{\partial F_2}{\partial y_1}\right) \neq 0.$$

that is,

$$\frac{\partial F_1}{\partial y_1}\frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2}\frac{\partial F_2}{\partial y_1} \neq 0.$$

This is a familiar formula, namely that for a determinant. Thus we have in fact assumed that

$$\begin{vmatrix} \frac{\partial F_1}{\partial y_1}, & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1}, & \frac{\partial F_2}{\partial y_2} \end{vmatrix} = \det\left(\frac{\partial F_i}{\partial y_j}\right)_{i,j} \neq 0.$$

(And this condition suffices: if we assume that this determinant is non-zero we have *either*

$$\frac{\partial F_2}{\partial y_2}(\mathbf{x}^0, y_1^0, y_2^0) \neq 0$$

and/or

$$\frac{\partial F_2}{\partial y_1}(\mathbf{x}^0, y_1^0, y_2^0) \neq 0,$$

so if the latter holds, we can start by solving $F_2(\mathbf{x}, y_1, y_2) = 0$ for y_1 instead of y_2.)

Details.

Text: Chapter XV, Sections 1,2 and 3