Repetition.

Total differential:

μ continuous in a neighborhood U of $\mathbf{o}$ such that $\mu(\mathbf{o}) = 0$,

and numbers $A_1, \ldots, A_n$ for which

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \sum_{k=1}^{n} A_k h_k + \|\mathbf{h}\|\mu(\mathbf{h})$$

Implies partial derivatives,

follows from <u>continuous</u> partial derivatives.

Interpretations:

Tangent hyperplane.

Aproximations.

Computing: arihtmetic rules quite like with plain derivatives,

Chain Rule. *Let* $f(\mathbf{x})$ *have a total differential in* $\mathbf{a}$*. Let* $g_k(t_1, \ldots, t_r)$ *have partial derivatives in* $\mathbf{b} = (b_1, \ldots, b_r)$ *and let* $g_k(\mathbf{b}) = a_k$ *for* $k = 1, \ldots, n$*. Then*

$$(f \circ \mathbf{g})(t_1, \ldots, t_r) = f(\mathbf{g}(\mathbf{t})) = f(g_1(\mathbf{t}t), \ldots, g_n(\mathbf{t}))$$

has all the p. d. in $\mathbf{b}$*, and one has*

$$\frac{\partial (f \circ \mathbf{g})(\mathbf{b})}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial f(\mathbf{a})}{\partial x_k} \cdot \frac{\partial g_k(\mathbf{b})}{\partial t_j}.$$

If we compose

$$\mathbb{E}_k \xrightarrow{\mathbf{g}} \mathbb{E}_n \xrightarrow{\mathbf{f}} \mathbb{E}_m$$

we obtain

$$D(\mathbf{f} \circ \mathbf{g}) = D\mathbf{f} \cdot D\mathbf{g}$$

where $D\mathbf{h} = \left(\frac{\partial h_i(\mathbf{a})}{\partial x_k}\right)_{ik}$

Lagrange in several variables.

Proposition. *Let f have continuous partial derivatives in a convex open set $U \subseteq \mathbb{E}_n$. Then for any two $x, y \in D$ there is a θ, $0 \leq \theta \leq 1$, such that*

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial f(\mathbf{x} + \theta(\mathbf{y} - \mathbf{x}))}{\partial x_j}(y_j - x_j).$$

Often encountered in the form

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial f(\mathbf{x} + \theta\mathbf{h})}{\partial x_j} h_j.$$

Partial derivative as a function

$$\frac{\partial f}{\partial x_k} : D' \to \mathbb{R},$$

higher order partial derivatives

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_k \partial x_l}, \quad \frac{\partial^r f(\mathbf{x})}{\partial x_{k_1} \partial x_{k_2} \dots \partial x_{k_r}},$$

partial derivatives of order r.

The order is determined by the number of taking derivatives

$$\frac{\partial^3 f(x,y,z)}{\partial x \partial y \partial z} \quad \text{a} \quad \frac{\partial^3 f(x,y,z)}{\partial x \partial x \partial x}$$

are derivatives of order 3.

Subsequent taken of derivative by the same variable is written as an exponent, e.g.

$$\frac{\partial^5 f(x,y)}{\partial x^2 \partial y^3} = \frac{\partial^5 f(x,y)}{\partial x \partial x \partial x \partial y \partial y}, \quad \frac{\partial^5 f(x,y)}{\partial x^2 \partial y^2 \partial x} = \frac{\partial^5 f(x,y)}{\partial x \partial x \partial y \partial y \partial x}.$$

Proposition. *Let partial derivatives* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *be defined and let them be continuous in a neighborhood of* (x, y)*. Then*

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}.$$

Theorem. *Let* f *in* n *variables have continuous partial derivatives up to order* k*. Then the values of the derivatives depend only on the numbers of taking derivatives in the individual variables .*

$$\frac{\partial^r f}{\partial x_1^{r_1} \partial x_2^{r_2} \dots \partial x_n^{r_n}} \quad \text{kde} \quad r_1 + r_2 + \cdots + r_n = r.$$

From preceding semester

In the sequel we will need more about metric spaces, in particular something on compactness and completeness. Recall the compact (that is, closed and bounded) intervals, and that

- each sequence in such $\langle a, b\rangle$ has a convergent subsequence,
- and that every continuous function on them attains a maximum and a minimum.

In more detail:

Supremum of a set M :

(1) for every $x \in M$, $x \leq s$, and

(2) if $x \leq y$ for every $x \in M$ then $s \leq y$.

For linear $\leq$ equivalent with

(1) for every $x \in M$, $x \leq s$, and

(2) if $y < s$ then there is an $x \in M$ such that $y < x$.

Theorem. *Each sequence on a compact interval contains a convergent subsequence.*

Explicitely: Let a, b *are reals such that* $a \leq x_n \leq b$ *for all* n*. Then there is a subsequence* $(x_{k_n})_n$ *of* $(x_n)_n$ *converging in* $\mathbb{R}$*, and we have* $a \leq \lim_n x_{k_n} \leq b$*.*

Proof. Set

$$M = \{x \mid x \in \mathbb{R},\ x \leq x_n \text{ for infinitely many } n\}.$$

M is non-void and bounded as $a \in M$ and b is an upper bound of M. thus there exists $s = \sup M$

a platí $a \leq s \leq b$.

$$M = \{x \,|\, x \in \mathbb{R},\ x \leq x_n \text{ for infinitely many } \ n\}.$$

For every n, the set

$$K(n) = \{k \,|\, s - \frac{1}{n} < x_k < s + \frac{1}{n}\}$$

is infinite: indeed, there is an $x > s - \varepsilon$ such that $x_n > x$ for infinitely many n, while by the definition of M there are only finitely many n such that

Pick a k_1 such that

$$s - 1 < x_{k_1} < s + 1.$$

Let $k_1 < k_2 < \cdots < k_n$ be chosen so that for $j = 1, \ldots, n$

$$s - \frac{1}{j} < x_{k_j} < s + \frac{1}{j}.$$

Since $K(n+1)$ is infinite, we can choose av$k_{n+1} > k_n$ such that

$$s - \frac{1}{n+1} < x_{k_{n+1}} < s + \frac{1}{n+1}.$$

Thus chosen subsequence $(x_{k_n})_n$ of $(x_n)_n$ obviously converges to s.

Compact spaces.

A metric space (X, d) is *compact* if every sequence in (X, d) contains a convergent subsequence.

Proposition. *A subspace of a compact space is compact iff it is closed.*

Proof. I. Let Y be a closed subspace of a compact X and let $(y_n)_n$ be a sequence in Y. As a sequence in X it has a subsequence with limit in X, and by closedness this limit is in Y.

II. If Y is not closed then there is a sequence $(y_n)_n$ in Y konvergent in X such that $y = \lim_n y_n \notin Y$. Then $(y_n)_n$ cannot have a subsequence convergent in Y because each subsequence converges to y.

Proposition. *Let (X,d) be arbitrary and let a subspace Y of X be compact. Then Y is closed in (X,d).*

Proof. Let $(y_n)_n$ in Y converge in X to y. Then every subsequence of $(y_n)_n$ converges to y and hence $y \in Y$.

A metric space (X,d) is *bounded* if we have for some K that

$$\forall x, y \in X, \quad d(x,y) < K.$$

Proposition. *Each compact space is bounded.*

Proof. Choose an arbitrary x_1 and x_n so that $d(x_1, x_n) > n$. The sequence $(x_n)_n$ has no convergent subsequence: if x were a limit of such subsequence there would be for a sufficiently large n infinitely many elements of this subsequence closer to x_1 then$d(x_1, x_n) + 1$, a contradiction.

Theorem. *A product of finitely many compact spaces is compact.*

Proof. Suffices for two spaces.

Let (X, d_1), (Y, d_2) be compact and let $((x_n, y_n))_n$ be a sequence in $X \times Y$. Choose a convergent subsequence $(x_{k_n})_n$ of $(x_n)_n$ and a convergent subsequence $(y_{k_{l_n}})_n$ of $(y_{k_n})_n$. Then

$$((x_{k_{l_n}}, y_{k_{l_n}}))_n$$

is a convergent subsequence of $((x_n, y_n))_n$.

A *compact interval* in $\mathbb{E}_n$ is a product of closed bounded intervals $\langle a_i, b_i \rangle$

Theorem. *A subspace of the euclidean space $\mathbb{E}_n$ is compact iff it is bounded and closed.*

Proof. I. We already know it has to be closed and bounded.

II. Now let $Y \subseteq \mathbb{E}_n$ be bounded and closed. Since it is bounded,

$$Y \subseteq J^n \subseteq \mathbb{E}_n$$

for a sufficiently large interval.

J^n is compact as a product of $\langle a_i, b_i \rangle$, and since Y is closed in $\mathbb{E}_n$ it is also closed in J^n and hence compact.

Proposition. *Let* $f : (X, d) \to (Y, d')$ *be a continuous mapping and let* $A \subseteq X$ *be compact. Then* $f[A]$ *is compact.*

Proof. Let $(y_n)_n$ be a sequence in $f[A]$. Choose $x_n \in A$ so that $y_n = f(x_n)$. Let $(x_{k_n})_n$ be convergent. Then $(y_{k_n})_n = (f(x_{k_n}))_n$ is convergent $(x_n)_n$.

Proposition. *Let* (X, d) *be compact. Then every continuous mapping* $f : (X, d) \to \mathbb{R}$ *attains a minimum and a maximum.*

Proof. $Y = f[X] \subseteq \mathbb{R}$ is compact. Hence it is bounded and has to have a supremum M and an infimum m. Obviously $d(m, Y) = d(M, Y) = 0$ and since Y is closed, $m, M \in Y$.

We know that a continuous f is characterized by closed *preimages* of closed sets. Now we see that if the domain is compact we also have that the *images* of closed subsets are closed. Consequenly we have (a.o.) the following.

Theorem. *If (X, d) is compact then each $f : (X, d) \to (Y, d')$ one-to one onto continuous map is a homeomorphism.*

More generally, *let $f : (X, d) \to (Y, d')$ and $g : (X, d) \to (Z, d'')$ a be continuous and let $h : (Y, d') \to (Z, d'')$ be such that $h \circ f = g$. Then h is continuous.*

Proof. Let B be closed in Z. Then $A = g^{-1}[B]$ is closed and hence compact in X and hence $f[A]$ is compact, and hence closed in Y. since f is onto, we have $f[f^{-1}[C]] = C$ for every C. Thus,

$$h^{-1}[B] = f[f^{-1}[h^{-1}[B]]] = f[(h \circ f)^{-1}[B]] = f[g^{-1}[B]] = f[A]$$

is closed.

A sequence $(x_n)_n$ in a metric space (X, d) is *Cauchy* if

$$\forall \varepsilon > 0 \ \exists n_0 \text{ such that } \ m, n \geq n_0 \ \Rightarrow \ d(x_m, x_n) < \varepsilon.$$

Observation. *Each convergent sequence is Cauchy.*

Proposition. *A Cauchy sequence with a convergent subsequence converges (and namely to the limit of the subsequence).*

Proof. Let $(x_n)_n$ be Cauchy and let $\lim_n x_{k_n} = x$. Let $d(x_m, x_n) < \varepsilon$ for $m, n \geq n_1$ and $d(x_{k_n}, x) \leq \varepsilon$ for $n \geq n_2$. If we set $n_0 = \max(n_1, n_2)$ we have for $n \geq n_0$ (since $k_n \geq n$)

$$d(x_n, x) \leq d(x_n, x_{k_n}) + d(x_{k_n}, x) < 2\varepsilon.$$

A metric space (X, d) is *complete* if each Cauchy sequence in (X, d) converges.

Notes. 1. By the Bolzano-Cauchy theorem, the Euclidean line $\mathbb{R}$ is complete.

2. Calculus is possible.

3. Strong equivalence of distances preserves the Cauchy property and completeness, the plain equivalence does not.

Proposition. *A subspace of a complete space is complete iff it is closed.*

Proof. I. Let $Y \subseteq (X, d)$ be closed. Let $(y_n)_n$ be Cauchy in Y. Then it is Cauchy and hence convergent in X, and by closedness the limit is in Y.

II. Let Y not be closed. Then there is a sequence $(y_n)_n$ in Y convergent in X such that $\lim_n y_n \notin Y$. Then $(y_n)_n$ is Cauchy in X, and since the distance is the same, also in Y. Bu it does not converge in Y.

Proposition. *Every compact space is complete.*

Proof. A Cauchy sequence has, by compactness, a convergent subsequence, and hence it converges.

Lemma. *A sequence*

$$(x_1^1, \ldots, x_n^1), (x_1^2, \ldots, x_n^2), \ldots, (x_1^k, \ldots, x_n^k), \ldots$$

is Cauchy in $\prod_{i=1}^n (X_i, d_i)$ *iff each* $(x_i^k)_k$ *is Cauchy in* (X_i, d_i).

Proof. $\Rightarrow$ follows immediately from the fact that $d_i(u_i, v_i) \leq d((u_j)_j, (v_j)_j)$.

$\Leftarrow$: Let each $(x_i^k)_k$ be Cauchy. For $\varepsilon > 0$ and i choose k_i such that for $k, l \geq k_i$, $d_i(x_i^k, x_i^l) < \varepsilon$. Then for $k, l \geq \max_i k_i$

$$d((x_1^k, \ldots, x_n^k), (x_1^l, \ldots, x_n^l)) < \varepsilon.$$

Theorem. *A product of complete spaces is complete. In particular* $\mathbb{E}_n$ *is complete.*

And from that immediately follows

Corollary. *A subspace* Y *of the Euclidean* $\mathbb{E}_n$ *is complete iff it is closed.*

Details.

Text: Chapter XIII, Sections 7 and 6