We will study real functions of several real variables

$$f(x_1, \ldots, x_n)$$

We already know quite a few facts about functons of one variable; thus, let us consider (fixing all the variables but one) instead

$$f(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots a_n)$$

a system of functions of one variables that we understand.

This will not work, though.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \text{ for } (x,y) \neq (0,0), \\ 0 \text{ for } (x,y) = (0,0). \end{cases}$$

$f(x,0) = 0, f(0,x) = 0$ are both nicely continuous

for $a \neq 0$ are $f(x,a)$ and $f(a,x)$ given on all the domain as simple arithmetic formulas

but $f(0,0) = 0$, and for arbitrarily small ε, hence arbitratily close to $(0,0)$ we have

$$f(\varepsilon, \varepsilon) = \frac{1}{2}$$

Metric space

$(X, d), \; d : X \times X \to \mathbb{R}$

$$d(x,y) \geq 0, \quad d(x,y) = 0 \Leftrightarrow x = y$$
$$d(x,y) = d(y,x)$$
$$d(x,z) \leq d(x,y) + d(y,z)$$

So far we worked in a special one

$$(\mathbb{R}, |x-y|)$$

Similarily we have another

$$(\mathbb{C}, |x-y|)$$

(**Caution**: triangle inequality for $|x-y|$ is not quite as trivial as in $\mathbb{R}$)

Euclidean spaces $\mathbb{E}_n$: $(\mathbb{R}^n, d)$

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_i (x_i - y_i)^2}$$

For our purposes it will be particularly important. You know it also from linear algebra in the form of the vector space $\mathbf{V}_n$ with scalar product $\mathbf{uv}$ and norm $||\mathbf{u}|| = \sqrt{\mathbf{uu}}$ – and distance $||\mathbf{u} - \mathbf{v}||$

(X, d) with $d(x, y) = 1$ for $x \neq y$ (discrete space)

$F(a, b)$ set of all bounded functions on an interval $\langle a, b \rangle$

$$d(f, g) = \sup\{|f(x) - g(x)| \mid a \leq x \leq b\}$$

Subspace. (X, y) m. space, $Y \subseteq X$

$$(Y, d') \quad \text{where} \quad d'(x, y) = d(x, y)$$

Continuous map(ping)s $f : (X, d) \to (Y, d')$

$$\forall x \in X, \forall \varepsilon > 0 \exists \delta > 0,$$
$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$$

(Compare with

$$\forall x \in X, \forall \varepsilon > 0 \exists \delta > 0,$$
$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

Trivialities: Identical map $\mathrm{id} : (X, d) \to (X, d)$,

Embedding of a subspace $j = (x \mapsto x) : (Y, d') \to (X, d)$

Composition $gf : (X, d) \to (Z, d'')$ of continuous maps $f : (X, d) \to (y, d')$ a $g : (Y, d') \to (Z, d'')$ is continuous

Convergence. $\lim_n x_n = x$:

$$\forall \varepsilon > 0 \exists n_0 (n \geq n_0 \Rightarrow d(x, x_n) < \varepsilon)$$

(Compare with

$$\forall \varepsilon > 0 \exists n_0 (n \geq n_0 \Rightarrow |x - x_n| < \varepsilon)$$

Theorem. *A mapping* $f : (X_1, d_1) \to (X_2, d_2)$ *is continuous iff for each convergent* $(x_n)_n$ *in* (X_1, d_1) *the sequence* $(f(x_n))_n$ *converges in* (X_2, d_2) *and it holds* $\lim_n f(x_n) = f(\lim_n x_n)$.

Proof. I. Let f be continuous, let $\lim_n x_n = x$. For $\varepsilon > 0$ choose, using continuity, $\delta > 0$ s.t. $d_2(f(y), f(x)) < \varepsilon$ for $d_1(x, y) < \delta$. By definition of convergence there is an n_0 s.t. for $n \geq n_0$ one has $d_1(x_n, x) < \delta$. Thus, if $n \leq n_0$ we have $d_2(f(x_n), f(x)) < \varepsilon$, and then $\lim_n f(x_n) = f(\lim_n x_n)$.

II. Let f not be continuous. Then there is an $x \in X_1$ and an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is an $x(\delta)$ with

$$d_1(x, x(\delta)) < \delta \quad \text{while} \quad d_2(f(x), f(x(\delta))) \geq \varepsilon_0.$$

Set $x_n = x(\frac{1}{n})$. Then $\lim_n x_n = x$ but $(f(x_n))_n$ cannot converge to $f(x)$. $\square$

Neighborhood. Set

$$\Omega(x,\varepsilon) = \{y \,|\, d(x,y) < \varepsilon\}$$

"U is a neighborhood of x"

$$\equiv \;\exists \varepsilon > 0, \Omega(x,\varepsilon) \subseteq U$$

Observation: 1. U nbh. of x and $U \subseteq V \;\Rightarrow V$ nbh. of x.

2. U,V nbh.of $x \;\Rightarrow U \cap V$ nbh.of x.

Open sets. $U \subseteq (X,d)$ is *open* if it is a neighborhood of each of its point.

Observation. *Each* $\Omega_X(x,\varepsilon)$ *is open in* (X,d).

Observation. $\emptyset$ *and* X *are open. If* U_i*,* $i \in J$*are open then* $\bigcup_{i \in J} U_i$ *is open, and if* U *and* V *are open so is* $U \cap V$.

Closed sets. $A \subseteq (X, d)$ is *closed* in $(X, d) \equiv$ for every $(x_n)_n \subseteq A$ convergent in X, $\lim_n x_n$ is in A.

Proposition. *$A \subseteq (X, d)$ is closed in (X, d) iff $X \smallsetminus A$ is open.*

Proof. I. Let $X \smallsetminus A$ not be open. Then there is an $x \in X \smallsetminus A$ such that for every n, $\Omega(x, \frac{1}{n}) \nsubseteq X \smallsetminus A$. Choose $x_n \in \Omega(x, \frac{1}{n}) \cap A$. Then $(x_n)_n \subseteq A$ and $\lim x_n = x \notin A$, and hence A is not closed.

II. Let $X \smallsetminus A$ be open and let $(x_n)_n \subseteq A$ converge to $x \in X \smallsetminus A$. Then we have for some $\varepsilon > 0$, $\Omega(x, \varepsilon) \subseteq X \smallsetminus A$ and hence for sufficiently large n, $x_n \in \Omega(x, \varepsilon) \subseteq X \smallsetminus A$– a contradiction. $\square$

Corollary. *$\emptyset$ and X are closed sets. If A_i, $i \in J$ are closed, $\bigcap_{i \in J} A_i$ is closed, and if A and B are closed then so is $A \cup B$.*

$d(x, A) = \inf\{d(x, a) \mid a \in A\}$.

Closure of a set A:

$$\overline{A} = \{x \mid d(x, A) = 0\}.$$

Proposition. (1) $\overline{\emptyset} = \emptyset$.

(2) $A \subseteq \overline{A}$,

(3) $A \subseteq B \quad \Rightarrow \quad \overline{A} \subseteq \overline{B}$,

(4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, a

(5) $\overline{\overline{A}} = \overline{A}$.

Proof. (4): $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$. If $x \in \overline{A \cup B}$ but not $x \in \overline{A}$ we have $\alpha = d(x, A) > 0$ and hence $y \in A \cup B$ with $d(x, y) < \alpha$ are in B; thus $x \in \overline{B}$.

(5): Let $d(x, \overline{A}) = 0$. Choose $\varepsilon > 0$. Then there is $z \in \overline{A}$ such that $d(x, z) < \frac{\varepsilon}{2}$ and hence for this z we can choose $y \in A$ such that $d(z, y) < \frac{\varepsilon}{2}$. Thus $d(x, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and we see that $x \in \overline{A}$. $\square$

Proposition. $\overline{A}$ *is the set of all limits of convergent* $(x_n)_n \subseteq A$.

Proposition. $\overline{A}$ *is closed, and it is the least closed set containing* A. *Hence,*

$$\overline{A} = \bigcap \{B \mid A \subseteq B,\ B \text{ closed}\}.$$

Proof. If $(x_n)_n \subseteq \overline{A}$ converges to x choose $y_n \in A$ with $d(x_n, y_n) < \frac{1}{n}$. Then $\lim_n y_n = x$ and x is in $\overline{A}$.

If B is closed and $A \subseteq B$ choose $x \in \overline{A}$ anconvergent $(x_n)_n$ in A, tedy v B, such that $\lim x_n = x$. Then $x \in B$. $\square$

Theorem. *Bud'te* $(X_1, d_1), (X_2.d_2)$ *metric spaces and* $f : X_1 \to X_2$ *a mapping Then the following are equivalent.*

(1) f *is continuous.*

(2) *For every* $x \in X_1$ *and every nbh* V *of* $f(x)$*there is a nbh* U *of* x *such that* $f[U] \subseteq V$.

(3) *For every* U *open in* X_2 *the preimage* $f^{-1}[U]$ *is open in* X_1.

(4) *For every* A *closed in* X_2 *the preimage* $f^{-1}[A]$ *is closed in* X_1.

(5) *For every* $A \subseteq X_1$ *we have* $f[\overline{A}] \subseteq \overline{f[A]}$.

A property or concept is *topological* if it is preserved by homeomorphisms. Thus e.g. the following are topological concepts:

- convergence
- openness
- closedness
- closure
- neighborhood
- or continuity itself.

Equivalent and strongly equivalent metrics:

The identical map $(X, d_1) \to (X, d_2)$ is a homeomorphism (*"d_1 and d_2 are equivalent"*).

d_1 a d_2 are *strongly equivalent* if there are positive constants α and β with

$$\alpha \cdot d_1(x, y) \leq d_2(x, y) \leq \beta \cdot d_1(x, y).$$

In the euclidean spaces where we had so far the distance

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

set

$$\lambda((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|, \quad \text{and}$$

$$\sigma((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_i |x_i - y_i|.$$

Proposition. *d, λ and σ are strongly equivalent metrics on $\mathbb{E}_n$.*

Products. For (X_1, d_i), $i = 1, \ldots, n$ define on the cartesian product $\prod_{i=1}^n X_i$ a metric

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_i d_i(x_i, y_i).$$

The resulting space

$$(\prod_{i=1}^n X_i, d) = \prod_{i=1}^n (X_i, d_i)$$

is called the product of the (X_i, d_i).

One also writes

$$(X_1, d_1) \times \cdots \times (X_n, d_n).$$

Hence

$$(\mathbb{E}_n, \sigma) = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n \text{ times}} = \mathbb{R}^n.$$

Theorem. 1. *The projections* $p_j = ((x_i)_i \mapsto x_j) : \prod_{i=1}^{n}(X_i, d_i) \to (X_j, d_j)$ *are continuous maps.*

2. *Let* $f_:(Y, d') \to (X_j, d_j)$ *be arbitrary continuous maps. Then the uniquely defined* $f : (Y, d') \to \prod_{i=1}^{n}(X_i, d_i)$ *satisfying* $p_j \circ f = f_j$*, that is, the map defined by* $f(y) = (f_1(y), \ldots, f_n(y))$*, is continuous.*

Hence, if we study maps

$$\mathbf{f} = (f_1, \ldots, f_m) : \mathbb{E}_n \to \mathbb{E}_m$$

the continuity in the range depends on the continuity in the individual coordinates only, unlike in the domain.