Topics. Thanks for the questions suggesting what topics I should return to in this and the January lectures. There is room for more.

So far I was asked to

(a) tell more about the basics of the Implicit Functions Theorem,

(b) explain what is the role in the implicit function theorem in the extremes with constraints,

(c) tell some more about the Lebesgue integral,

(d) from the tutorials: discuss some points of metric spaces.

(e) I would also like to return to series (you had Taylor series). If there will be time I will explain whether, when and why

$$\sum_{n=1}^{\infty} a_n$$

can be viewed as a sum of infinitely many summands (extending the concept of finite sums of indexed summands) and when it is just the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_n$$

(a) Implicit Functions Theorems. Given $F^i(x_1, \ldots, x_m, y_1, \ldots, y_n)$, $i = 1, \ldots, n$ functions in $n+m$ variables and we wish to solve the system of equations

$$
\begin{aligned}
F^1(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0, \\
\ldots \quad \ldots \quad \ldots \\
F^n(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0.
\end{aligned}
$$

In what sense?

Rewrite it as follows

$$
\begin{aligned}
F^1_{x_1, \ldots, x_m}(y_1, \ldots, y_n) &= 0, \\
\ldots \quad \ldots \quad \ldots \\
F^n_{x_1, \ldots, x_m}(y_1, \ldots, y_n) &= 0.
\end{aligned}
$$

Just a more suggestive terminology: we will speak of the variables $y_1, \ldots, y_n$ as of *unknowns*.

Thus we have for each vector $\mathbf{x} = (x_1, \ldots, x_n)$ a system of

n equations in n unknowns $y_1, \ldots, y_n$

$$F^1_{\mathbf{x}}(y_1, \ldots, y_n) = 0,$$
$$\ldots \quad \ldots \quad \ldots$$
$$F^n_{\mathbf{x}}(y_1, \ldots, y_n) = 0$$

and it should have under reasonable conditions (recall n *linear* equations in n unknowns) a unique solution

$$y_{1\mathbf{x}}, \ldots, y_{n\mathbf{x}}.$$

With changing $\mathbf{x}$ we have changing systems of equations and hence of course, changing solutions $y_{k\mathbf{x}}$. Thus we obtain *functions in m variables*

$$f_k(x_1, \ldots, x_m) = f_k(\mathbf{x}) = y_{k\mathbf{x}}$$

and Implicit Functions Theorem speaks about these functions.

Recall linear algebra: a system of linear equations

$$
\begin{aligned}
L^1(y_1, \dots, y_n) &= 0, \\
\dots \quad \dots \quad \dots \\
L^n(y_1, \dots, y_n) &= 0
\end{aligned}
$$

that is,

$$
\begin{aligned}
\sum_{j=1}^{n} A_{1j} y_j - b_1 &= 0 \\
\dots \quad \dots \quad \dots \\
\sum_{j=1}^{n} A_{nj} y_j - b_n &= 0,
\end{aligned}
$$

has a unique solution iff

$$\det A_{ij} \neq 0.$$

Now remember that if our functions $F_k(x_1, \ldots, x_m, y_1, \ldots, y_n)$ have continuous partial derivatives, they have total differentials, and the functions $F^k_{\mathbf{x}}(y_1, \ldots, y_n)$ are in a (small) neighborhood of a point of a solution $\mathbf{y}^0$ well approximated by linear

$$\sum_{j=1}^{n} A_{kj}(y_j - y^0_j) + F^{\cdots}_j(\mathbf{y}^0) = \sum_{j=1}^{n} A_{kj} y_j + B_j$$

with $A_{kj} = \frac{\partial F_k}{\partial y_j}$.

Thus it should not come as a surprise that the condition for unique solution is expressed by non-zero Jacobi determinant

$$\det\left(\frac{\partial F_k}{\partial y_j}\right).$$

(b) The role of IFT in extremes with constraints.

Situation: We seek local extremes of a function

$$f(x_1, \ldots, x_n)$$

defined on a domain D. No problem in the interior of D: there they are among the points with all partial derivatives zero. Trouble with the extremes that are on the border on D.

Suppose the border of D expressed by conditions (constraints)

$$g_i(x_1, \ldots, x_n) = 0, \; i = 1, \ldots, k.$$

The theorem solving the problem,

Theorem. *Let* $f, g_1, \dots, g_k$ *be real functions defined in an open set* $D \subseteq \mathbb{E}_n$*, and let them have continuous partial derivatives. Suppose that the rank of the matrix*

$$M = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1}, & \dots, & \dfrac{\partial g_1}{\partial x_n} \\ \dots, & \dots, & \dots \\ \dfrac{\partial g_k}{\partial x_1}, & \dots, & \dfrac{\partial g_k}{\partial x_n} \end{pmatrix}$$

is the largest possible, that is k*, everywhere in* D*.*

If the function f *achieves at a point* $\mathbf{a} = (a_1, \dots, a_n)$ *a local extreme subject to the constraints*

$$g_i(x_1, \dots, x_n) = 0, \quad i = 1, \dots, k$$

then there exist numbers $\lambda_1, \dots, \lambda_k$ *such that for each* $i = 1, \dots, n$ *we have*

$$\frac{\partial f(\mathbf{a})}{\partial x_i} + \sum_{j=1}^{k} \lambda_j \cdot \frac{\partial g_j(\mathbf{a})}{\partial x_i} = 0.$$

is subjected to the condition that the rank of the matrix

$$M = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}, & \ldots, & \frac{\partial g_1}{\partial x_n} \\ \ldots, & \ldots, & \ldots \\ \frac{\partial g_k}{\partial x_1}, & \ldots, & \frac{\partial g_k}{\partial x_n} \end{pmatrix}$$

is maximal possible, that is, k (we have $k < n$). Why is that, and how it helps?

The matrix has to have a regular square $k \times k$ submatrix, without loss of generality let

$$\det \begin{pmatrix} \frac{\partial g_1}{\partial x_1}, & \ldots, & \frac{\partial g_1}{\partial x_k} \\ \ldots, & \ldots, & \ldots \\ \frac{\partial g_k}{\partial x_1}, & \ldots, & \frac{\partial g_k}{\partial x_k} \end{pmatrix} \neq 0 \qquad (*)$$

The constraints constitute a system of equations

$$g_1(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = 0,$$
$$\ldots \quad \ldots \quad \ldots$$
$$g_k(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = 0.$$

with unknowns $x_1, \ldots, x_k$ and parameters (variables) $x_{k+1}, \ldots, x_n$.

Now $(*)$ is the Jacobi determinant condition allowing us to use the IFT to obtain $x_j = \phi_j(x_{k+1}, \ldots, x_n)$ and we can compute the
extremes of f on the border of D
as extremes of the function

$$F(x_{k+1}, \ldots, x_n) =$$
$$= f(\phi_1(x_{k+1}, \ldots, x_n), \ldots, \phi_k(x_{k+1}, \ldots, x_n), x_{k+1}, \ldots, x_n)$$

with $x_{k+1}, \ldots, x_n$ in an open set, hence looking for the zero partial derivatives (of the function F).

(c) Lebesgue integral.

This was just information, without construction or proofs.

The technique of Riemann integral can be extended so that we can keep everything we are able to do, without changes. Moreover, the following reckoning, easy to remember, is safe.

If $\int_{D_n} f$ *exist for* $n = 1, 2, ...,$ *then* $\int_{\bigcup_n D_n} f$ *exists.*

Let D *be bounded and* $|f_n(x)| \le C$ *for a fixed* C. *Let* $\lim_n f_n$ *exist.*

If $\int_D f_n$ *exist for* $n = 1, 2, ...,$ *then* $\int_D \lim_n f_n$ *exists and we have*

$$\int_D \lim_n f_n = \lim_n \int_D f_n.$$

These are rules very easy to remember. One has much more, for instance

(1) If $\int_D f_n$ exist and $(f_n)_n$ is monotone then $\int_D \lim_n f_n = \lim_n \int_D f_n$.

(2) If on the right hand side everything makes sense and and $|f(t,x)| \leq g(x)$ fot an integrable g then

$$\int_D f(t_0,x)\mathrm{d}x = \lim_{t\to t_0} \int_D f(t,x)\mathrm{d}x.$$

(3) If

$$\left|\frac{\partial f(t,x)}{\partial t}\right| \leq g(x).$$

and everything makes sense in a neighborhood U of t_0 then

$$\int_D \frac{\partial f(t_0,-)}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_D f(t_0,-).$$

(e) Infinite sums of reals.

Series (you had Taylor series). Given a sequence $(a_n)_n$ of real numbers one speaks of the associated *series* as of making sense to a "sum" $\sum_{n=1}^{\infty} a_n$. The basic definition is the limit

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_n.$$

Here is an example showing that we may have troubles viewing it as a sum of the a_n involved. Consider the series with $a_n = (-1)^{n+1}\frac{1}{n}$, that is,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Obviously $\lim_{n\to\infty} \sum_{k=1}^{n} a_n$ exists (it is somewhere between $\frac{1}{2}$ and 1).

Divide the a_n into two sets

$$1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7}, \dots, \text{and} \qquad (*)$$
$$-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \dots.$$

It is easy to see that the sets $\{\sum_{k=1}^{n} a_{2n-1} \mid n = 1, 2, \dots\}$ and $\{\sum_{k=1}^{n} a_{2n} \mid n = 1, 2, \dots\}$ are both unbounded.

Try to add the series in a new order as follows. First add the elements of the upper part in $(*)$ until we get at least 10, then follow with elements of the lower part in $(*)$ untill we are under 5; then follow in the upper part untill get more than 100, continue in the lower part untill we get under 50, etc.. In this reordering α of the sequence the limit $\lim_{n\to\infty} \sum_{k=1}^{n} a_{\alpha(n)}$ is infinite!

Note that in this manner one can reorder the series to *any* sum.

Thus, we can hardly think of the limit of partial sums as of a sum of the system of numbers.

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A sum s of an infinite sequence $(a_n)_n$ should be well approximated by finite sums $\sum_{k \in K} a_k$, $K \subseteq \mathbb{N}$ finite. This is so in case of the so called absolutely convergent series.

A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem. *Let $s = \sum_{n=1}^{\infty} a_n$ converge absolutely. Then for every $\varepsilon > 0$ there is a finite subset $K \subseteq \mathbb{N}$ such that for any finite L with $K \subseteq L \subseteq \mathbb{N}$, $|s - \sum_L a_n| < \varepsilon$ (in other words, such that for each $M \subseteq \mathbb{N}$ disjoint with K, $|\sum_M a_n| < \varepsilon$).*

Proof. $(\sum_{k=1}^{n} |a_k|)_n$ is a Cauchy sequence. Let n_0 be such that for all $m > n \geq n_0$, $\sum_{k=1}^{m} |a_k| - \sum_{k=1}^{n} |a_k| < \varepsilon$. We have $\sum_{k=1}^{m} |a_k| - \sum_{k=1}^{n} |a_k| = \sum_{k=n+1}^{m} |a_k|$ and for every M finite disjoint with $K = \{1, 2, \ldots, n_0\}$ we have an m such that $M \subseteq \{n = n_0 + 1, \ldots m\}$ and

$$|\sum_{M} a_k| \leq \sum_{M} |a_k| \leq \sum_{k=n_0+1}^{m} |a_k| < \varepsilon.$$