Repetition.

Fubini Theorem. *Interval* $J = J' \times J''$ $J' \subseteq \mathbb{E}_m$, $J'' \subseteq \mathbb{E}_n$, $\quad f : J \to \mathbb{R}$.

$$\int_J f(\mathbf{x},\mathbf{y})\, d\mathbf{xy} = \int_{J'} (\int_{J''} f(\mathbf{x},\mathbf{y})\, d\mathbf{y})\, d\mathbf{x}$$

Thus in particular in two variables

$$\int_J f = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y) \mathrm{d}y \right) \mathrm{d}x,$$

and generally

$$\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\cdots \left(\int_{a_n}^{b_n} f(x_1, x_2 \ldots, x_n) \mathrm{d}x_n \right) \cdots \right) \mathrm{d}x_2 \right) \mathrm{d}x_1$$

so that we can compute multivariable integral using primitive functions.

Problem: Like in other parts of multivariable calculus, we have a problem with <u>domains of functions</u> $f : D \to \mathbb{R}$.

While in one variable, a compact interval $\langle a, b\rangle$ is a quite common domain, starting with $\mathbb{E}_2$ the n-dimensional intervals (bricks) are very special and one would wish (at least, for integration satisfactory) compact domains D.

Compacts subspaces of $\mathbb{E}_n$ are precisely the bounded closed subsets, hence one can approach the problem by

- first embedding the D into a brick J,
- and then extending f by values 0 on $J \setminus D$.

Note that for the integration purposes it does not matter which brick containing D we choose.

But we have to recall also the assumptions of Fubini theorem, so far not remembered. One assumes the existence of $\int_J f$, and it is not obvious that for (say) a continuous f the function thus extended (typically badly discontinuous on the border of D) qualifies.

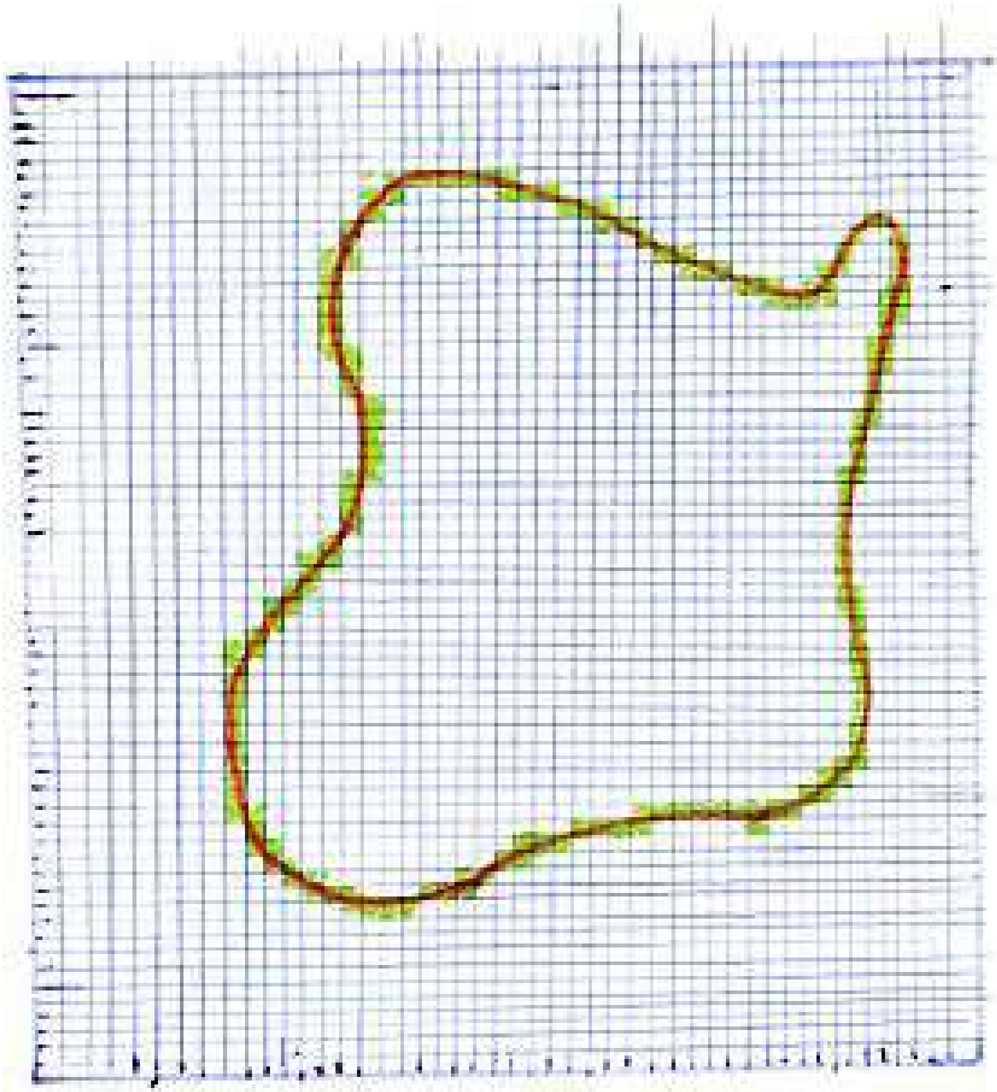
Intuitively: The volume of the border Δ of D is typically 0. The volume of the union of the bricks of a partition P meeting the border is diminishing with diminishing mesh and is smaller than a given $\varepsilon > 0$ for sufficiently fine P. Then, in the lower and upper sums the contribution of the the brick meeting Δ

$$\sum\{m(f,B)\cdot \mathsf{vol}(B)\,|\,B\in\mathcal{B}(P), B\cap\Delta\neq\emptyset\},$$
$$\sum\{M(f,B)\cdot \mathsf{vol}(B)\,|\,B\in\mathcal{B}(P), B\cap\Delta\neq\emptyset\}$$

in

$$s(f,P)=\sum\{m(f,B)\cdot \mathsf{vol}(B)\,|\,B\in\mathcal{B}(P)\},$$
$$S(f,P)=\sum\{M(f,B)\cdot \mathsf{vol}(B)\,|\,B\in\mathcal{B}(P)\}$$

is negligible.



Lebesgue integral (information, no construction, no proofs).

Riemann integral is intuitively very satisfactory and does what one wishes to be done – when it works.

- But it may not exist for quite natural functions, or at least it is not easy to see whether it exists or not,
- and we cannot perform useful operations (limiting, taking derivatives) universally enough.

(In the latter, it is not that it would give wrong values – those it yields are correct; but it may not yield any.)

Lebesgue integral is an extension of Riemann integral where we can do practically everything
under conditions that are very easy to remember.

Some Lebesgue integration rules.

(In (3)-(7) existence on the left implied.)

(1) If J is an interval (brick) and Riemann $\int_J f$ exists, then it coincides with the Lebesgue one.

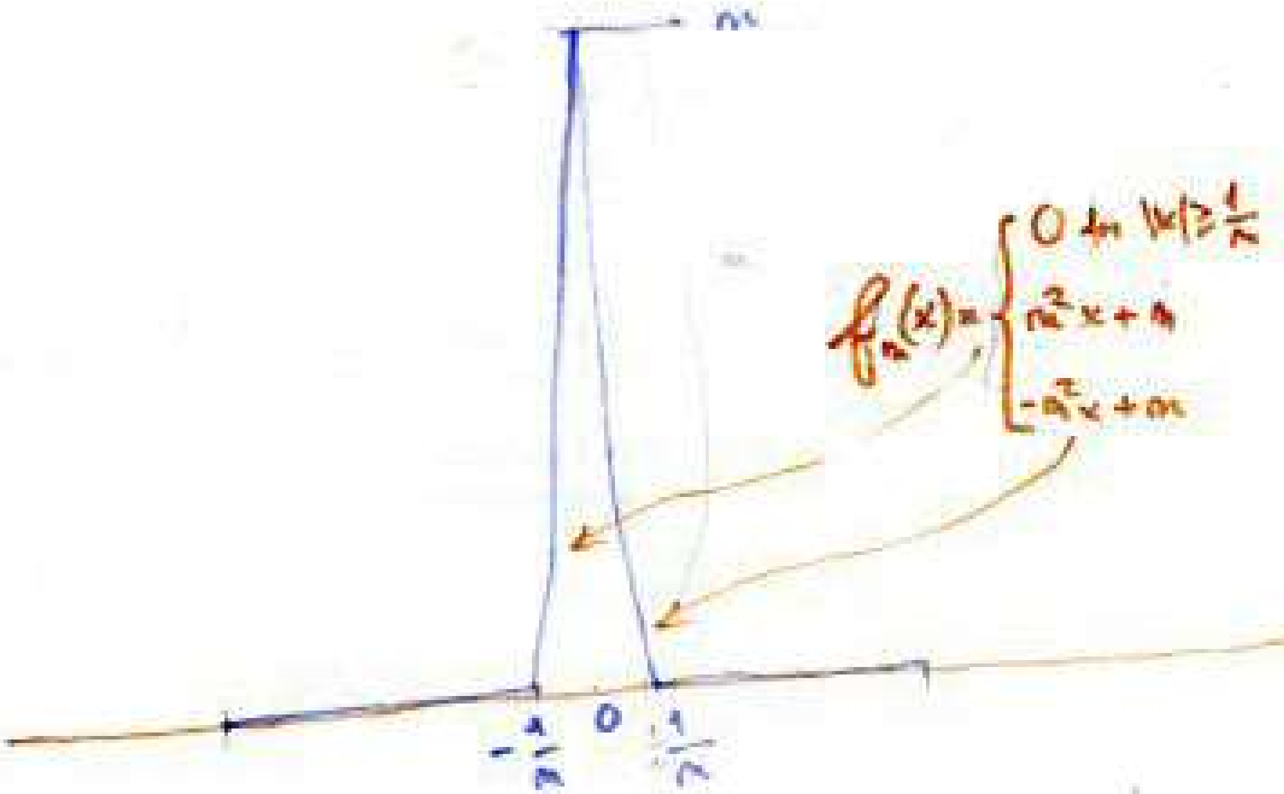
(2) If $\int_{D_n} f$ exist for $n = 1, 2, \ldots$ then $\int_{\bigcup D_n} f$ exists

(3) If $\int_D f_n$ exist and $(f_n)_n$ is monotone then $\int_D \lim_n f_n = \lim_n \int_D f_n$.

(4) If $\int_D f_n$ exist and $|f_n| \leq g$ for some g such that $\int_D g$ exists then $\int_D \lim_n f_n = \lim_n \int_D f_n$.

(5) (A practical consequence of (4)) If D is bounded, $|f_n(x)| \leq C$ and $\int_D f_n$ exist then

$$\int_D \lim_n f_n = \lim_n \int_D f_n.$$



$$g_n(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ n & \text{for } x = 0 \end{cases}$$

$$\int f_n(x)\,dx = 1 \qquad \int g_n(x)\,dx = 0$$

$$\neq \quad \lim f_n = \lim g_n \quad \neq$$

$$\circ \qquad \qquad \circ$$

And some more:

(6) Let U be a neighborhood of t_0 and g such that $\int_D g$ exists and $\int_D f(t,x)\mathrm{d}x$ exist and $|f(t,x)| \leq g(x)$ for all $t \in U \smallsetminus \{t_0\}$ then

$$\int_D f(t_0,x)\mathrm{d}x = \lim_{t\to t_0} \int_D f(t,x)\mathrm{d}x.$$

(7) If

$$\left|\frac{\partial f(t,x)}{\partial t}\right| \leq g(x).$$

and everything makes sense in a neighborhood U of t_0 then

$$\int_D \frac{\partial f(t_0,-)}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_D f(t_0,-).$$

Note. Very roughly: Riemann integral was based on thinking of volumes that add (with correction) for *finite* unions. Recalling absolutely convergent series one realizes that *countable* sums make perfect sense. One of the approaches to Lebesgue integral is in analyzing volumes of countable unions.

A somewhat surprising example indicating that it is a quite substantial change. In the unit interval $\mathbb{I} = \langle 0, 1 \rangle$ order all the rational numbers into a sequence

$$r_1, r_2, \dots, r_n \dots$$

and consider open intervals $U_n = (r_n - \frac{\varepsilon}{2^n}, r_n + \frac{\varepsilon}{2^n})$. The union $\bigcup_{n=1}^{\infty} U_n$ is dense in $\mathbb{I}$ and the lengths of the constituting intervals add to a volume

$$< 4\varepsilon$$

while the lengths of fintely many open intervals constituting a dense subset of $\mathbb{I}$ always add to 1!

Integral on compact D as indicated a week ago makes sense.

First let us learn, without proof, a useful

Theorem. (Tietze) *Let Y be closed in a metric space X. Then each continuous real function f on Y such that $a \leq f(x) \leq b$ for all x can be extended to an equally bounded continuous g on X.*

We had a bounded function

$$f : D \to \mathbb{R}$$

defined on a compact $D \subseteq \mathbb{E}_n$. We choose a brick $J \supseteq D$, defined

$$\widetilde{f} : J \to \mathbb{R}$$

by setting

$$\widetilde{f}(x) = \begin{cases} f(x) \text{ for } x \in D, \\ 0 \text{ for } x \in J \setminus D \end{cases}$$

and try to define

$$\int_D f = \int_J \widetilde{f}.$$

Problem. If, say, f is continuous, does $\int_J \widetilde{f}$ exist?

What we can do: Obviously $\phi = (x \mapsto d(x, D)) : J \to \mathbb{R}$ is continuous, hence

$$J_n = \{x \,|\, d(x, D) \geq \frac{1}{n}\} = \phi^{-1}[\langle \frac{1}{n}, +\infty)]$$

is closed and $J_n \cap D = \emptyset$. Hence, obviously, $f_n : J_n \cap D \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} f(x) & \text{for } \ x \in D \\ 0 \ \ \text{for } \ x \in J_n \end{cases}$$

is continuous and we can extend it to equally bounded continuous g_n on J by Tietze Theorem. Then

$$\lim g_n = \widetilde{f}$$

and the desired integral exists by the Lebesgue rule (5) above.

A closed

$$f_n^{-1}[A] = \begin{cases} f^{-1}[A] & A \not\ni 0 \\ f^{-1}[A] \cup J_n & A \ni 0 \end{cases}$$

closed

Substitution.

Let ϕ be an increasing function with derivative defined on a neigborhood of a compact interval $\langle a, b\rangle$ mapping it onto $\langle \phi(a), \phi(b)\rangle$. Let f be a continuous function, and let F be a primitive function of f. Then for $G = f \circ \phi$ we have

$$G'(x) = F'(\phi(x))\phi'(x)$$

and hence by the (consequence of the) Fundamental Theorem of Calculus

$$\int_{\phi(a)}^{\phi(b)} f(x)\mathrm{d}x = F(\phi(b)) - F(\phi(a)) =$$

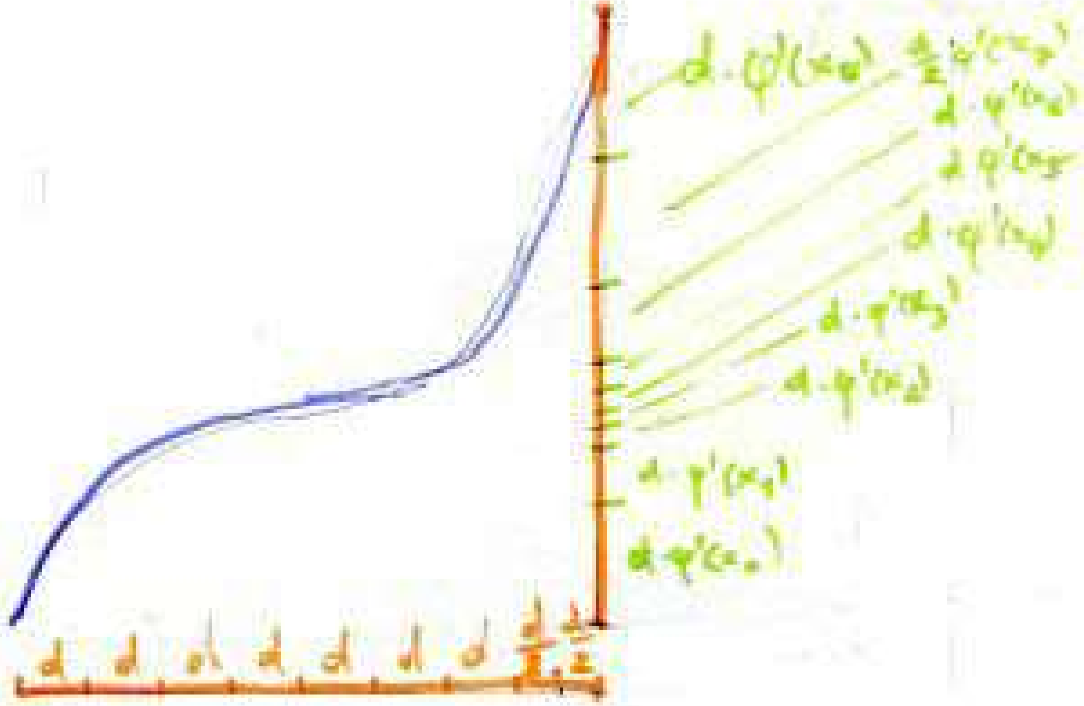
$$= G(b) - G(a) = \int_a^b f(\phi(x))\phi'(x)\mathrm{d}x.$$

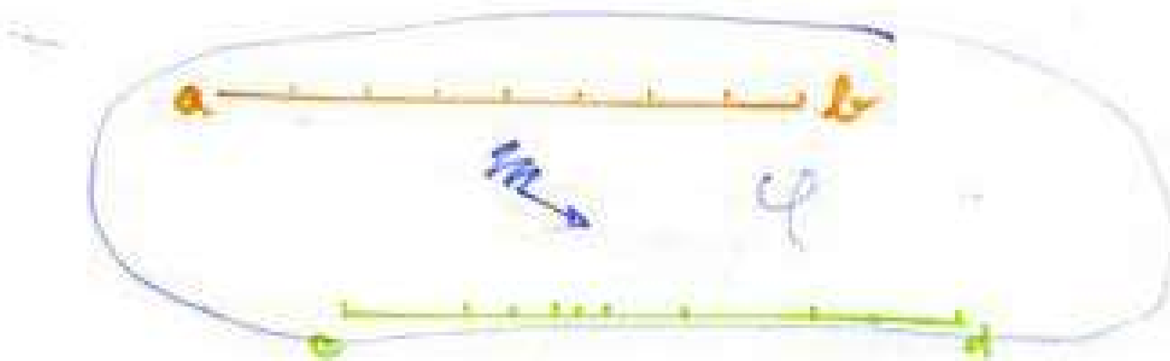
The resulting rule

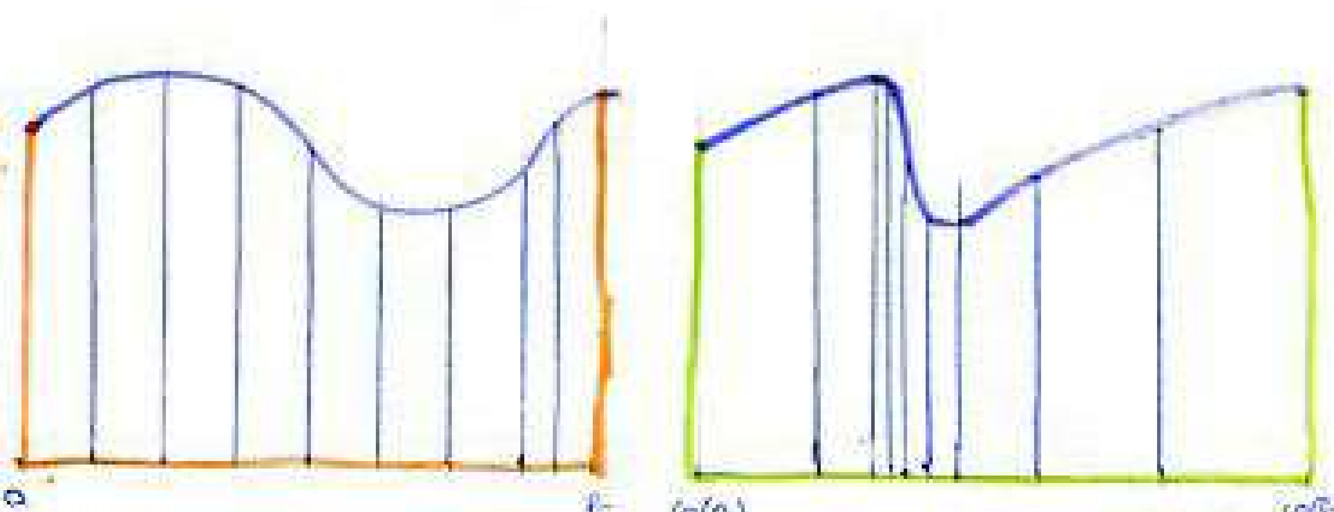
$$\int_{\langle \phi(a),\phi(b)\rangle} f(x)\mathrm{d}x = \int_{\langle a,b\rangle} f(\phi(x))\phi'(x)\mathrm{d}x$$

has a clear geometric interpretation.

The increasing $\phi : \langle a, b\rangle \to \langle \phi(a), \phi(b)\rangle$ describes a deformation of $\langle a, b\rangle$ stretching resp. compressing small subintervals $\langle x, x+h\rangle$ in the rate of approximately $\phi'(x)$ (the interval of length h is deformed by the mean value theorem to one of length $\phi'(x+\theta h)h$, approximately $\phi'(x)h$). If we compute the integral of a function f over the deformed interval then in the the associated integral *before* the deformation, the summands of rectangles with bases $\langle x, x+h\rangle$ correspond to rectangles with length of bases (approx.) $h \cdot \phi'(x)$ – see picture. Thus the $\phi'(x)$ in the formula is a compensation for the local deformation of the basis at x.




a
b
φ
c
d


a
b
φ(a)
φ(b)

Substitution in multivalued integral.

Suppose we have a compact $D \subseteq \mathbb{E}_n$ and a one-to-one regular map $\phi : U \to \mathbb{E}_n$ with $D \subseteq U$. Recall the Jacobian

$$\frac{\mathsf{D}(\phi)}{\mathsf{D}(\mathbf{x})} = \det \left(\frac{\partial \phi_i}{\partial x_j} \right)_{i,j=1,\ldots,m}.$$

The substitution formula for the integral over $\phi[D]$ is

$$\int_{\phi[D]} \mathbf{f} = \int_D \mathbf{f}(\phi(\mathbf{x})) \left| \frac{\mathsf{D}(\phi)}{\mathsf{D}(\mathbf{x})} \right| \mathrm{d}\mathbf{x}$$

Note: Concerns Lebesgue integral, and holds for much more general D and much more general $\mathbf{f}$.

Instead of proof. Recall the compensation of the deformation in one variable given by $\phi'(x)$. Here, a small cube vith volume h^n

$$\langle x_1, x_1{+}h\rangle \times \langle x_2, x_2{+}h\rangle \times \cdots \times \langle x_n, x_n{+}h\rangle$$

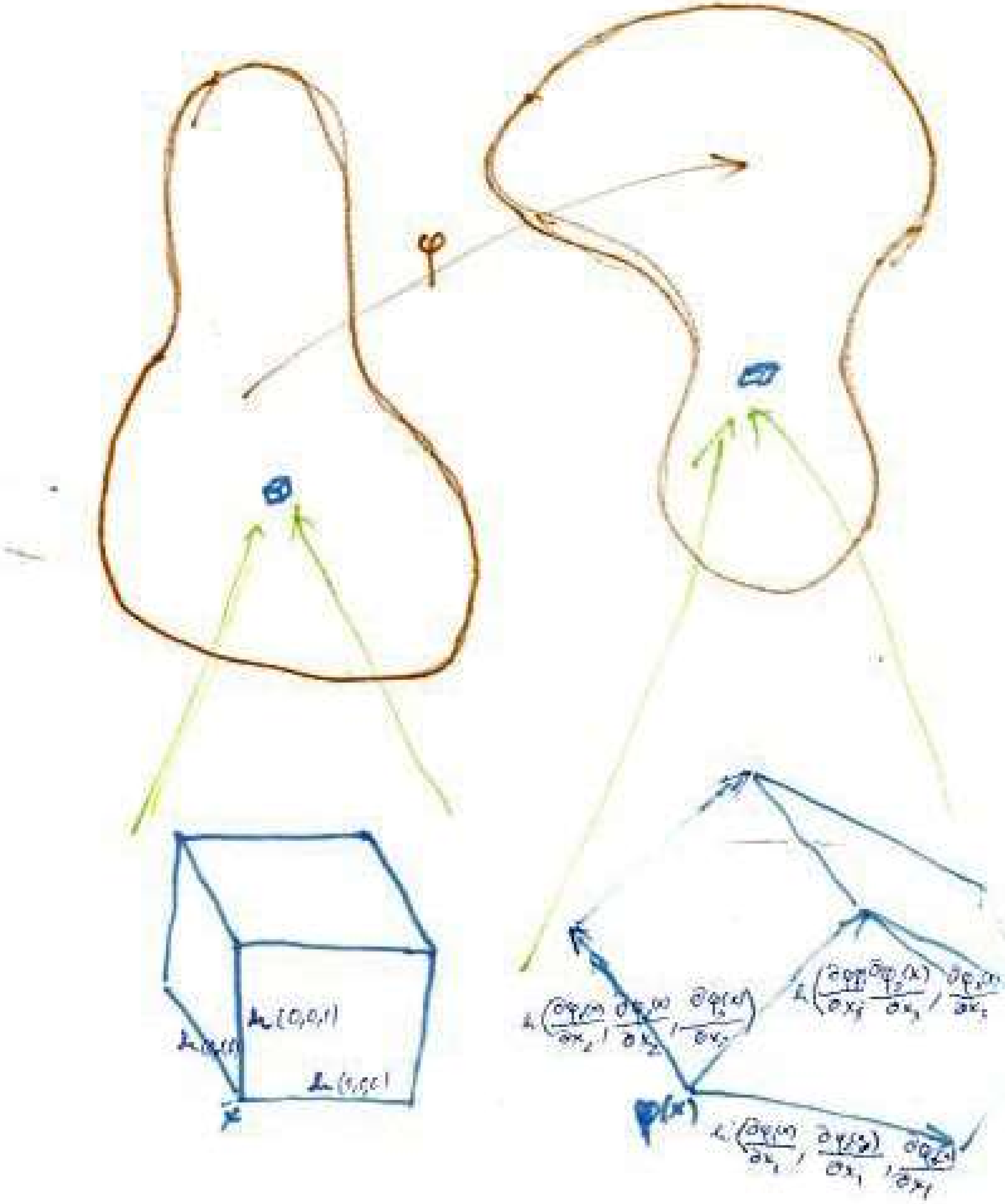
is deformed, approximately, into the paralleipiped determined by the vectors

$$\phi(\mathbf{x}) + h\cdot \left(\frac{\partial \phi_1(\mathbf{x})}{\partial x_1}, \frac{\partial \phi_2(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial \phi_n(\mathbf{x})}{\partial x_i}\right)$$
$$(i = 1, \ldots, n)$$

with volume

$$h^n \cdot \left| \frac{\mathsf{D}(\phi(\mathbf{x}))}{\mathsf{D}(\mathbf{x})} \right|$$

(see picture)


φ
h(0,0,1)
φ(x)

Details.

Text: Chapter XIV, 5; Chapter X, 4

Tietze.pdf