

Exercises for Combinatorial and Computational Geometry

Series 6 — bonus problems

deadline: 5.2.2025

Please submit the solutions of problems 1,2,3,6 separately from problems 4,5; each subset will be handled by a different corrector.

1. Let \mathcal{C} be the set of all cells (faces of dimension 2) in an arrangement of n lines in the plane. We denote the number of vertices of a cell C by $f_0(C)$. Prove that $\sum_{C \in \mathcal{C}} f_0(C)^2 = O(n^2)$. [2]
2. Let S be a set of n geometric objects in the plane. The *intersection graph* of S is a graph on n vertices that correspond to the objects in S . Two vertices are connected by an edge if and only if the corresponding objects intersect.

(a) The total number of all graphs on n given vertices is $2^{\binom{n}{2}} = 2^{n^2/2+O(n)}$. Prove that the total number of all intersection graphs of n line segments in the plane is only $2^{O(n \log n)}$. (Be careful and consider also collinear line segments!) Use the theorem about the number of sign patterns. [3]

(b) Show that the number of intersection graphs of n simple curves in the plane is at least $2^{\Omega(n^2)}$. If you wish, you can solve this exercise for n convex sets instead of simple curves. [2]

3. Let $P = \{p_1, p_2, \dots, p_n\}$ be a set of n points in the plane. We say that points x, y have the *same view* of P if the points of P are visible in the same cyclic order from x and y . That is, if we rotate light rays that emanate from x and y , respectively, the points of P are lit in the same order by these rays. We assume that neither x nor y is in P and that neither of them can see two points of P in occlusion.

Show that there exists a point set P such that there are $\Omega(n^4)$ other points in the plane with mutually distinct views of P . [3]

4. (a) Show that for every positive irrational number α there are infinitely many pairs of numbers $m, n \in \mathbb{N}$ such that

$$\left| \alpha - \frac{m}{n} \right| < \frac{1}{n^2}.$$

Use Theorem 2.1.3 from the lecture notes. [1]

- (b) Prove that for $\alpha = \sqrt{2}$ there are only finitely many pairs $m, n \in \mathbb{N}$ that satisfy

$$\left| \alpha - \frac{m}{n} \right| < \frac{1}{4n^2}. \quad [2]$$

- (c) Let α_1, α_2 be real numbers. Prove that for every $N \in \mathbb{N}$ there exist $m_1, m_2 \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \leq N$ such that for every $i \in \{1, 2\}$, we have

$$\left| \alpha_i - \frac{m_i}{n} \right| < \frac{1}{n\sqrt{N}}. \quad [2]$$

5. A point set P *pierces the triangles* of a point set M if every triangle determined by three points of M contains at least one point of P in its interior.

- (a) Prove that for every $n \geq 3$ and every n -point set $M \subset \mathbb{R}^2$ in general position there is a set P of $2n - 5$ points that pierces the triangles of M . [2]
- (b) For every $n \geq 3$, construct an n -point set $M \subset \mathbb{R}^2$ in general position such that no set P of $2n - 6$ points pierces the triangles of M . [2]
6. How many cells are there in the arrangement of $3\binom{d}{2}$ hyperplanes in \mathbb{R}^d with equations $x_i - x_j = -1, 0, 1$, where $1 \leq i < j \leq d$? [3]