Exercises for Combinatorial and Computational Geometry  
Series 3 — Crossing numbers and incidences  
deadline 29. 11. 2019

1. Prove that a graph with \( n \) vertices that has a rectilinear drawing in the plane with no three pairwise crossing edges has \( O(n^{3/2}) \) edges. You may use the crossing lemma. (A rectilinear drawing is a drawing where every edge is drawn as a straight-line segment.)

2. Let \( I_{\text{circ}}(n, m) \) be the maximum number of incidences of \( n \) points and \( m \) unit circles in the plane. Show that \( I_{\text{circ}}(n, n) = O(n^{4/3}) \).

3. Let \( \mathcal{M} = \{M_1, M_2, \ldots, M_n\} \) be a system of subsets of an \( n \)-element set \( N \) (that is, \( \forall i \in [n] \ M_i \subseteq N \) such that every pair of sets \( M_i, M_j \) has at most one common element. The number of incidences of \( N \) and \( \mathcal{M} \) is defined as \( I(N, \mathcal{M}) := \sum_{i=1}^{n} |M_i| \). Determine whether necessarily \( I(N, \mathcal{M}) = O(n^{4/3}) \). [2]

4. Find an \( n \)-point set in \( \mathbb{R}^4 \) with \( \Omega(n^2) \) unit distances.

5. Let \( P \) be an \( n \)-point set in the plane.
   
   (a) Let \( k > 1 \). Show that there are at most \( O(n^2/k^3 + n/k) \) lines such that each of them contains at least \( k \) points of \( P \), and that the number of incidences of these lines with \( P \) is at most \( O(n^2/k^2 + n) \). [3]
   
   (b) Let \( \alpha \in (0, \pi) \). Show that \( P \) determines at most \( O(n^{7/3}) \) triangles with at least one angle of size \( \alpha \). (Hint: split the triangles \( ABC \) with angle \( \alpha \) at \( A \) into two groups according to whether the line \( AC \) contains more than \( n^{1/3} \) points of \( P \).) [3]

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