Exercises for Combinatorial and Computational Geometry
Series 1 — Convex sets
hints 25.10.2019, deadline 1.11.2019

Please choose a nickname that will be used in the list of scores on the webpage of the exercises. If you have already chosen a nickname, you can just sign your solutions with either your name or your nickname.

1. Find a set $M \subset \mathbb{R}^2$ that is a union of two convex sets such that $\mathbb{R}^2 \setminus M$ consists of five pairwise disjoint connected components. \[2\]

2. Prove Carathéodory’s theorem (you may use Radon’s theorem or a part of its proof). \[2\]

3. Let $M = \{x_1, y_1, x_2, y_2, \ldots, x_{d+1}, y_{d+1}\}$ be a set of $2d + 2$ points in $\mathbb{R}^d$. Prove that $M$ can be partitioned into two subsets $A$ and $B$ such that each of these subsets contains, for every $i = 1, 2, \ldots, d + 1$, exactly one point from $\{x_i, y_i\}$, and the convex hulls of $A$ and $B$ have a nonempty intersection. (You may use the fact that the $(d+1)$-tuple of vectors $x_i - y_i$ is linearly dependent and then use an approach similar to the proof of Radon’s theorem.) \[2\]

4. Let $M$ be a finite set of at least four points in the plane such that each point is either red or blue. In addition, for every 4-tuple $V$ of points of $M$ there is a line strictly separating the red points of $V$ from the blue points of $V$. Prove that there is a line strictly separating all the red points of $M$ from all the blue points of $M$. \[3\]

5. Let $X_1, X_2, \ldots, X_{d+1}$ be finite point sets in $\mathbb{R}^d$ such that for every $i \in \{1, 2, \ldots, d+1\}$ the origin lies in conv($X_i$). Prove that there exist $d + 1$ points $x_1, x_2, \ldots, x_{d+1}$, with $x_i \in X_i$, such that conv($\{x_1, x_2, \ldots, x_{d+1}\}$) contains the origin. \[4, \text{hint}\]