

# Exercises for Combinatorial and Computational Geometry II

## Series 4 — Intersection patterns of convex sets

deadline 19. 5. 2020

- (Carathéodory's theorem, cone version.)  
Let  $A \subseteq \mathbb{R}^d$  and  $x \in \text{cone}(A)$ ; that is,  $x$  can be expressed as  $x = \sum_{i \in [n]} \alpha_i a_i$  for some  $n \in \mathbb{N}$ ,  $a_i \in A$  and  $\alpha_i \geq 0$ . Prove that there is a subset  $B \subseteq A$  of at most  $d$  points such that  $x \in \text{cone}(B)$ . [2]
- (Colorful Carathéodory's theorem, cone version.)  
Let  $x \in \mathbb{R}^d \setminus \{0\}$  and  $A_1, \dots, A_d \subseteq \mathbb{R}^d$  such that for every  $i \in [d]$ , we have  $x \in \text{cone}(A_i)$ . Prove that there are points  $a_1, \dots, a_d$  with  $a_i \in A_i$  such that  $x \in \text{cone}(\{a_1, \dots, a_d\})$ . [2]
- (Colorful Carathéodory's theorem, stronger version in the plane.)  
Let  $X_1, X_2, X_3$  be finite point sets in the plane such that for every  $i, j \in \{1, 2, 3\}, i \neq j$ , we have  $0 \in \text{conv}(X_i \cup X_j)$ . Prove that there are points  $x_1, x_2, x_3$  with  $x_i \in X_i$  such that  $0 \in \text{conv}(\{x_1, x_2, x_3\})$ . [3]
- (Helly's theorem, colored version.)  
Let  $\mathcal{C}_1, \dots, \mathcal{C}_{d+1}$  be finite families of convex sets in  $\mathbb{R}^d$  such that for every choice  $C_i \in \mathcal{C}_i, i \in [d+1]$ , we have  $\bigcap_{i \in [d+1]} C_i \neq \emptyset$ . Prove that there exists  $i \in [d+1]$  such that  $\bigcap \mathcal{C}_i \neq \emptyset$  (that is, there is a point common to all sets of certain color). Hint: consider lexicographic minima of the intersections and show that one of them is the desired point. [3]
- Prove that the quantity  $(r-1)(d+1)+1$  in Tverberg's theorem cannot be improved: for every  $d, r \geq 2$ , there are  $(r-1)(d+1)$  points in  $\mathbb{R}^d$  with no Tverberg  $r$ -partition. [2]
- Let  $S \subset \mathbb{R}^d$  be a set of  $d+2$  points such that  $S \cup \{0\}$  is in general position and  $0 \in \text{conv}(S)$ . Prove that there are exactly two  $(d+1)$ -tuples of points from  $S$  that contain 0 in their convex hull. [2]
- Let  $S$  be a set of  $n$  points in the plane in general position and let  $x \in \text{conv}(S)$ . Prove that there are at least  $n-2$  triangles with vertices in  $S$  that contain  $x$  in their convex hull. [2]