Exercises for Combinatorial and Computational Geometry II

Series 3 — Davenport–Schinzel sequences

deadline 28. 4. 2020

By $\lambda_s(n)$ we denote the maximum length of a Davenport–Schinzel sequence of order s over the symbols $1, 2, 3, \dots, n$. The *complexity of a cell* of an arrangement of geometric objects in the plane is the number of vertices or edges lying on the boundary of the cell, counted with multiplicity.

1. Prove that the number of Davenport–Schinzel sequences of order 2 over the alphabet $\{1, \dots, n\}$, of length $2n-1$, and such that the leftmost occurrences of symbols form an increasing sequence, is equal to the Catalan number C_{n-1}. *Catalan numbers* are defined as follows: $C_0 = 1$ and $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$ pro $n \geq 1$. For example, for $n = 3$ we have two such sequences: 12321 and 12131. **[2]**

2. *Complexity of a cell in an arrangement of segments*
   Let C be a cell in an arrangement of n segments in general position in the plane such that the union of the segments is connected.
   (a) We label the segments by $1, 2, \dots, n$. We walk along the boundary of C starting from a random point and write a sequence of labels of segments visited during the walk. Prove that such a sequence does not contain *ababab* as a subsequence, and so the compexity of the cell C is $O(\lambda_4(n))$. **[2]**
   (b) Prove that the compexity of the cell C is at most $O(\lambda_3(n))$. Hint: assign several different symbols to each segment. **[2]**

3. *The zone theorem via Davenport–Schinzel sequences*
   The zone of a line p in an arrangement of lines is the set of all faces (of all dimensions) visible from p. Prove by a reduction to Davenport–Schinzel sequences that the complexity of the zone of a line in an arrangement of n lines in the plane is at most $O(n)$. **[3]**

4. Let $g_1, g_2, \dots, g_m$ be graphs of m cotinuous piecewise-linear functions from $\mathbb{R}$ to $\mathbb{R}$ that are formed by n segments and rays in total. Prove that the complexity of the lower envelope of $g_1, g_2, \dots, g_m$ is $O(\frac{n}{m}\lambda_3(2m))$. In particular, for $m = O(1)$, the complexity of the lower envelope is linear. **[2]**

5. We define the matrices

$$N = \begin{pmatrix} * & 1 & * & 1 \\ 1 & * & 1 & * \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & * \\ 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & * & 1 \\ 1 & 1 & * \end{pmatrix}$$

   where $*$ stands for an arbitrary element. Let $A \in \{0,1\}^{n \times n}$ be an $n \times n$ matrix with entries 0 or 1. We say that A *avoids* N if there are no indices $i_1 < i_2$ and $j_1 < j_2 < j_3 < j_4$ such that $a_{i_2j_1} = a_{i_1j_2} = a_{i_2j_3} = a_{i_1j_4} = 1$. Similarly we define avoiding of the other matrices.
   (a) Prove that if A avoids N, then the number of 1-entries in A is at most $\lambda_3(n) + O(n)$. **[2]**
   (b) Prove that if A avoids L, then the number of 1-entries in A is at most $O(n)$. **[1]**
   (c) Find a matrix A avoiding U and containing at least $\Omega(n\log(n))$ 1-entries. **[2]**