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Graph recurrence

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Abstract

For a graph G, a *graph recurrence sequence* $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ of vectors is defined by the recurrence

$$\mathbf{x}_{t+1} = A\mathbf{x}_t, \qquad t = 0, 1, \ldots,$$

where A is the adjacency matrix of G and $\mathbf{x}_0$ is an initial vector. Each vector in this sequence can be thought of as a vertex labeling of G, the label at a given vertex at step $t+1$ obtained by summing the values at the adjacent vertices at step t. Based on graphical sequences, three concepts are defined: (1) for a graph to be *determined by a set of vectors*, (2) for two graphs to be *m-equivalent*, and (3) for the vertices of the graph to be *separated by a set of vectors*. Results concerning these notions are given, relations to the graph isomorphism problem are discussed, and numerous open problems are posed.

1. Introduction

Throughout this paper $[n] = \{1, 2, \ldots, n\}$ and G is a simple connected graph with vertex set $[n]$. If A is the adjacency matrix of G and $\mathbf{x}_0$ is a vector in $\mathbb{R}^n$, define a *graph recurrence sequence*

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$$

by the recurrence

$$\mathbf{x}_{t+1} = A\mathbf{x}_t, \qquad t = 0, 1, \ldots. \tag{1.1}$$

Each vector in this sequence can be thought of as a vertex labeling of G. The label at a vertex i is the value of the ith coordinate $\mathbf{x}(i)$ of $\mathbf{x}$. The label at a given vertex at step $t+1$ is obtained by summing the values at the adjacent vertices at step t. In Fig. 1 the initial vector is $\mathbf{x}_0 = (1, 0, 0, 0)$, and the first few terms are $\mathbf{x}_1 = (0, 1, 1, 0)$, $\mathbf{x}_2 = (2, 1, 1, 2)$, $\mathbf{x}_3 = (2, 5, 5, 2)$. If the initial vector for an arbitrary graph is $\mathbf{x}_0 = (1, 1, \ldots, 1)$, then the first term $\mathbf{x}_1$ gives the vertex degrees of the graph. The broad question is, given an initial vector

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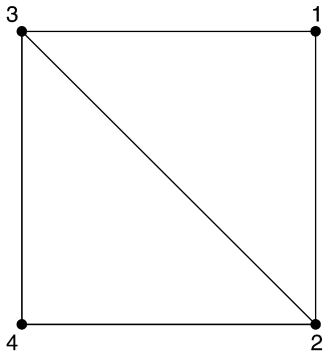

Fig. 1. Graph recurrence sequence: (1, 0, 0, 0), (0, 1, 1, 0), (2, 1, 1, 2), (2, 5, 5, 2), ...

$\mathbf{x}_0$ (or a set of initial vectors), what information about the graph can be inferred from its graph recurrence sequence.

Our initial motivation for investigating graph recurrences comes from the graph isomorphism problem: given two graphs (in terms of their adjacency matrices for example), to determine whether or not they are isomorphic. There are polynomial time algorithms for graph isomorphism in the case of interval graphs [7], planar graphs [6], in fact graphs of bounded genus [5], graphs of bounded degree [8], and graphs of bounded eigenvalue multiplicity [2]. In general, however, this problem holds a special place in algorithmic complexity theory because it remains open whether graph isomorphism is P or NP-complete or neither. The intent of this paper is not to settle this question, but to introduce a point of view and pose several questions. We show in Section 2 that a graphical sequence for a graph on n vertices is determined by the first $n + 1$ terms. The computation of these terms is algorithmically straightforward. Clearly isomorphic graphs produce the same graphical sequences up to a permutation of the coordinates. The question is to what extent the converse is true.

Three concepts are introduced in Sections 2–4, respectively. The first is for a graph G to be *determined by a graph recurrence sequence* (or graph recurrence sequences). Precise definitions of the three concepts appear in the respective sections, but basically for a graph G to be determined by a graph recurrence sequence means that G is the unique graph having that graph recurrence sequence. The *standard vector* $\mathbf{e}_i$ is a vector with 1 at coordinate i and all other coordinates 0. The graph recurrence sequence whose initial vector is $\mathbf{e}_i$ will simply be referred to as the graph recurrence sequence *centered at vertex* i. The graph recurrence sequence in Fig. 1, for example, is centered at vertex 1. If a graph G is determined by the graphical sequence centered at vertex i, then we say that G is *determined by vertex* i. For example, the graph in Fig. 1 is determined by vertex 1; it is easy to show that it is the unique graph with the sequence given in the caption of Fig. 1. Many graphs, for example complete graphs, complete bipartite graphs, cycles, wheels and trees, are determined by a single vertex. This is also the case if the vectors in the graph recurrence sequence span $\mathbb{R}^n$. In these

cases G can be distinguished from any other graph in polynomial time. Several open questions concerning which graphs are determined by one, or a fixed number, of vertices appear at the end of Section 2. In fact, numerous open problems appear throughout the paper.

The second concept is for two graphs to be *equivalent*. Basically this means that there is a "fake isomorphism" between the two graphs, a bijection between the vertex sets such that graphical sequences centered at corresponding vertices are identical (up to a permutation of the vertices). Equivalence of graphs (and more generally a stronger notion called *m-equivalence*) can be tested in polynomial time as a function of the number of vertices. An algorithm to do this, based on bipartite matching, is given in Section 3. Also in Section 3 an example is provided of a pair of non-isomorphic graphs that are 2-equivalent. Thus 2-equivalence is not a valid test for graph isomorphism.

We are unable, however, to provide an example of two non-isomorphic graphs that are 3-equivalent. In an attempt to provide such an example, the notion of m-regular graphs, a generalization of strongly regular graphs, is defined in Section 4. It is proved that a pair of m-regular graphs with the same set of parameters are m-equivalent. So the existence of a pair of non-isomorphic 3-regular graphs with the same set of parameters would also be an example of a non-isomorphic, 3-equivalent pair of graphs. However, for $m = 3$ and 4, a non-isomorphic pair of m-regular graphs with the same set of parameters is elusive, and for $m \geq 5$, no such pair can exist.

The third concept is for a graph to be *separated by a vertex* i (or set I of vertices). Basically this means that, for any pair of vertices, the values at the two vertices differ at some term in the graph recurrence sequence centered at vertex i (or some vertex $i \in I$). For the collection of graphs that can be separated by a single (or fixed number) of vertices, the graph isomorphism problem has a polynomial time solution. To what degree the vertices of a graph can be separated by a set of vertices is discussed in Section 5.

There is an extensive literature on the graph isomorphism problem. Although we are not aware of other papers using graph recurrence sequences, there are some similarities with known heuristics. For example, a common paradigm for heuristics is that of partitioning the vertices and refining the partition. Certainly our third concept, separating the set of vertices using graph recurrence sequences, can be put into that framework. Also it was recently pointed out that our notion of equivalence is closely related to a clever vertex labeling algorithm of Corneil and Gotlieb [4], although they do not use linear algebraic techniques. It has long been known that strongly regular graphs are particularly troublesome with respect to graph isomorphism; so it is not surprising to also find m-regular graphs in the paper cited above and in a paper of Cameron [3].

There are several questions posed at the end of each of Sections 2–5. It is our hope that the concepts introduced in this paper lead to interesting future work.

2. Determined graphs

Hereafter G_A will denote the graph whose adjacency matrix is A. The notation $\approx$ is used for graph isomorphism. Graph G_B is said to have the *same X-sequences* as graph G_A if, for some reordering of the vertices of B, the graph recurrence sequence (1.1) for

G_B and G_A are identical for all initial values in X. More precisely, there exists a single permutation matrix P such that

$$A^t\mathbf{x} = (P^{-1}BP)^t\mathbf{x}$$

for all $t \geq 0$ and for all $\mathbf{x} \in X$. A set $X \subset \mathbb{R}^n$ is said to *distinguish* a graph G_A from a graph G_B if G_B does not have the same X-sequences as graph G_A. Note that this is not a symmetric relation: X distinguishing G_A from G_B does not necessarily imply that X distinguishes G_B from G_A. A set $X \subset \mathbb{R}^n$ is said to *determine* a graph G_A if X distinguishes G_A from any graph not isomorphic to G_A.

A graph recurrence sequence is an infinite sequence. Lemma 2.1, however, implies that just the first $n+1$ terms are sufficient when considering whether a graph is determined by a set of vectors.

Lemma 2.1. *Let A and B be $n \times n$ matrices and $\mathbf{x} \in \mathbb{R}^n$. If $A^t\mathbf{x} = B^t\mathbf{x}$ for $n \geq t \geq 0$, then $A^t\mathbf{x} = B^t\mathbf{x}$ for all $t \geq 0$.*

Proof. The proof is by induction on t. Since $A^t\mathbf{x} = B^t\mathbf{x}$ for $n \geq t \geq 0$, also $A^t\mathbf{x} = B^t\mathbf{x}$ for $m \geq t \geq 0$, where m is the degree of the minimal polynomial for B. Now assume that $A^t\mathbf{x} = B^t\mathbf{x}$ for $k \geq t \geq 0$. After reducing by the minimal polynomial we have $B^k = g(B)$ where $\deg(g) < m$. Now $A^{k+1}\mathbf{x} = AA^k\mathbf{x} = AB^k\mathbf{x} = Ag(B)\mathbf{x} = Ag(A)\mathbf{x} = Bg(B)\mathbf{x} = B^{k+1}\mathbf{x}$. The second to last inequality follows because $\deg[xg(x)] \leq m$. □

Remark 2.2. It is an immediate consequence of Lemma 2.1 and the Cayley–Hamilton theorem that if A and B have the same characteristic polynomial and if $A^t\mathbf{x} = B^t\mathbf{x}$ for $n > t \geq 0$, then $A^t\mathbf{x} = B^t\mathbf{x}$ for all $t \geq 0$.

Theorem 2.3. *Let G_A be a graph and $X \subset \mathbb{R}^n$. If $\{A^t\mathbf{x} \mid \mathbf{x} \in X,\ t = 0, 1, \ldots\}$ spans $\mathbb{R}^n$, then G_A is determined by X.*

Proof. Assume that $A^t\mathbf{x} = (P^{-1}BP)^t\mathbf{x}$ for $t \geq 0$ and for all $\mathbf{x} \in X$. Then $A(A^t\mathbf{x}) = A^{t+1}\mathbf{x} = (P^{-1}BP)^{t+1}\mathbf{x} = (P^{-1}BP)(P^{-1}B^tP)\mathbf{x} = (P^{-1}BP)A^t\mathbf{x}$. Since $\{A^t\mathbf{x} \mid t \geq 0, \mathbf{x} \in X\}$ spans $\mathbb{R}^n$ we have $A\mathbf{x} = P^{-1}BP\mathbf{x}$ for all $x \in \mathbb{R}^n$. Therefore $A = P^{-1}BP$ and $G_A \approx G_B$. □

That the terms in the graph recurrence sequence span $\mathbb{R}^n$ is a sufficient, but not a necessary, condition for a vector to determine a graph. Consider the graph in Fig. 1. If $\mathbf{x}_0 = (1, 1, 0, 0)$, then the first four terms in the recurrence are $(1, 1, 0, 0)$, $(1, 1, 2, 1)$, $(3, 4, 3, 3)$, $(7, 9, 10, 7)$, which span $\mathbb{R}^4$. Therefore $\mathbf{x}_0$ determines the graph. Note, however, that $\mathbf{x}_0 = (1, 0, 0, 0)$ also determines this graph, but the terms in the graph recurrence sequence do not span $\mathbb{R}^4$.

Theorem 2.4. *Any graph is determined by a single vector.*

Proof. Let G_A be a graph on n vertices; take the vertex set to be $\{0, 1, 2, \ldots, n-1\}$. Let $\mathbf{x}_0 = (1, 2, 4, \ldots, 2^{n-1})$ and $\mathbf{x}_1 = A\mathbf{x}_0 = (a_1, \ldots, a_n)$. If $a_i = \sum_{j \in J} 2^j$ is the unique

base 2 representation of a_i, then vertex i must be adjacent to exactly the vertices in J. Therefore $\mathbf{x}_0$ determines G. □

Remark 2.5 (Isomorphism testing). Theorem 2.4 is unsatisfactory from an algorithmic point of view. The theorem shows that a given graph G_A is determined by the vector $\mathbf{x}_0 = (1, 2, 4, \ldots, 2^{n-1})$. Suppose we would like to test whether another graph, say G_B, is isomorphic to G_A. This involves computing the graph recurrence sequences $B^t(\mathbf{y}_0)$, $t = 0, 1, \ldots$ where, in the worst case, $\mathbf{y}_0$ ranges over all vectors obtained by permuting the coordinates of $\mathbf{x}_0$. Computing each sequence can be done efficiently, but there are $n!$ such permutations of the coordinates of $\mathbf{x}_0$, an intractable situation. If, on the other hand, G_A is determined by a single standard vector $\mathbf{e}$, then there are only n permutations of the coordinates of $\mathbf{e}$ to check. It is for this reason that, in the remainder of the paper, the initial vectors are always taken to be standard vectors.

We say that a graph G is determined by a set U of vertices if G is determined by the corresponding set $\{\mathbf{e}_i \mid i \in U\}$ of standard vectors.

Theorem 2.6. *The complete graphs K_n, complete bipartite graphs $K_{m,n}$, cycles C_n, wheels W_n and trees are determined by a single vertex.*

Proof. We prove the result for trees and leave the other more routine cases as exercises, noting that the initial vector for the wheel should have coordinate 1 at a vertex other than the hub.

Consider tree T rooted at vertex 1, and let $\mathbf{x}_0 = (1, 0, \ldots, 0)$. The *depth* of a vertex is its distance from the root and is denoted $d(i)$. It is easy to show by induction that, for t odd,

$$\mathbf{x}_t(i) = \begin{cases} 0 & \text{if } d(i) \text{ is even or } d(i) > t \\ 1 & \text{if } d(i) = t \\ >0 & \text{if } d(i) \text{ is odd and } d(i) < t \end{cases}$$

and, for t even,

$$\mathbf{x}_t(i) = \begin{cases} 0 & \text{if } d(i) \text{ is odd or } d(i) > t \\ 1 & \text{if } d(i) = t \\ >0 & \text{if } d(i) \text{ is even and } d(i) < t. \end{cases}$$

Now let G be a graph with the same graph recurrence sequence as T. Assign a *level*

$$l(i) = \min\{t \mid \mathbf{x}_t(i) \neq 0\}$$

to each vertex i of G. By the formula above for the graph recurrence sequence, it is clear that for $k \geq 0$ we have: (1) no two level k vertices are adjacent and (2) each vertex at level $k+1$ is adjacent to a unique vertex at level k. Therefore G must be a tree rooted at vertex 1 and $l(i) = d(i)$ for all i.

Assume that to depth k the graphs G and T are isomorphic. This is certainly true for $k = 0, 1$. Then G and T will be isomorphic to depth $k+1$ if the degrees of the vertices at depth k are the same in G and T. Let i be a vertex at depth k. Then $\mathbf{x}_{k+2}(i) = \deg(i) + \mathbf{x}_{k+1}(p(i))$

where $p(i)$ is the parent of vertex i. Therefore $\deg(i) = \mathbf{x}_{k+2}(i) - \mathbf{x}_{k+1}(p(i))$ is determined by just the graph recurrence sequence, the same values for G and T. □

Example 2.7 (A graph not determined by m particular vertices). To say that a graph G is determined by a single vertex is not to say that G is determined by any single vertex. It is easy to give, for any positive integer m, an example of a graph and a particular set U of m of its vertices such that G is not determined by U. Let H and H' be two non-isomorphic graphs with the same number of vertices and both regular of the same degree. Let G be the graph obtained by joining each vertex of the complete graph K_m to each vertex of H. Similarly, let G' be the graph obtained by joining each vertex of K_m to each vertex of H'. Then G and G' have the same graph recurrence sequences for each vertex in K_m. Therefore G is not determined by the set of vertices in K_m.

Theorem 2.6 lists some graphs that are determined by a single vertex. Examples are given in Sections 3 and 4 of graphs that are not determined by any vertex, or even any two vertices.

Questions 2.8. Several open questions naturally arise.

1. Classify the graphs determined by a single standard vector.
2. Does there exist a natural number m such that any connected graph is determined by some m vertices?
3. If the answer to Question 2 is no, then are all graphs with maximum degree m determined by m vertices?
4. If the eigenvalues of a graph are distinct, then is the graph determined by a single vertex? Are graphs with maximum eigenvalue multiplicity m determined by m vertices?
5. How many vertices suffice to determine any connected planar graph?

3. Equivalent graphs

Consider the action of a permutation $g : [n] \to [n]$ on a vector $\mathbf{a} = (a_1, \ldots, a_n)$ defined by $g(\mathbf{a}) = (a_{g1}, \ldots, a_{gn})$. Call two sequences of vectors

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$$
$$\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N)$$

equivalent, denoted $\mathbf{X} \equiv \mathbf{Y}$, if there exists a single permutation g such that $\mathbf{y}_i = g(\mathbf{x}_i)$ for all i. Given a graph G_A on n vertices we will use the notation

$$\mathbf{A}_i = \{A^0\mathbf{e}_i, A^1\mathbf{e}_i, A^2\mathbf{e}_i, \ldots, A^n\mathbf{e}_i\}$$

for the graph recurrence sequence centered at vertex i. A pairs of graphs G_A and G_B on n vertices will be called *equivalent*, denoted $G_A \equiv G_B$, if there exists a bijection $f : [n] \to [n]$ such that

$$\mathbf{A}_i \equiv \mathbf{B}_{f(i)}$$

for all $i \in [n]$. This means that there is a "fake isomorphism", a bijection between the vertex sets of the two graphs such that the graph recurrence sequences centered at corresponding vertices are the same. Although $\mathbf{A}_i$ is a finite sequence, Lemma 2.1 insures equivalence of the corresponding infinite graph recurrence sequences. Clearly $G_A \approx G_B$ implies that $G_A \equiv G_B$.

Theorem 3.1. *Equivalence of n vertex graphs can be tested in time polynomial in n.*

The proof of the theorem uses the following algorithm.

Algorithm 1.

Input Sequences $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ and $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N)$ of vectors in $\mathbb{R}^n$.

Output Whether or not $\mathbf{X} \equiv \mathbf{Y}$. Moreover, if it exists, a permutation g such that $g(\mathbf{x}_i) = \mathbf{y}_i$ for all i.

1. Denoting $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $\mathbf{y}_i = (y_{i1}, y_{i2}, \ldots, y_{in})$, let $\overline{\mathbf{x}}_i = (x_{1i}, x_{2i}, \ldots, x_{Ni})$ and $\overline{\mathbf{y}}_i = (y_{1i}, y_{2i}, \ldots, y_{Ni})$.
2. Rearrange $(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \ldots, \overline{\mathbf{x}}_n)$ and $(\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2, \ldots, \overline{\mathbf{y}}_n)$ in lexicographic order, with $g_{\mathbf{x}}$ and $g_{\mathbf{y}}$ the permutations that realize the respective sortings. (The lexicographic order is with respect to the usual order on the real numbers.)
3. Compare the sorted lists to determine whether they are identical. If they are, then $g = g_{\mathbf{y}}^{-1} \circ g_{\mathbf{x}}^{-1}$ is the required permutation.

Since only sorting and comparing corresponding elements are involved, Algorithm 1 is clearly polynomial. The algorithm that validates Theorem 3.1 is now as follows:

Algorithm 2.

Input Graphs G_A and G_B on n vertices.

Output Whether or not $G_A \equiv G_B$.

1. Let $M_i = \{j \in [n] \mid \mathbf{A}_i \equiv \mathbf{B}_j\}$. The set M_i is determined by n repetitions of Algorithm 1.
2. A bijection $f : [n] \to [n]$ such that $\mathbf{A}_k \equiv \mathbf{B}_{f(k)}$ for every k is given by a system of distinct representatives of $M_1, M_2, \ldots, M_n$. This is equivalent to finding a perfect matching in the bipartite graph, where one partite set is $[n]$, the other partite set is $\{M_1, M_2, \ldots, M_n\}$, and vertex i is adjacent to vertex M_j if $i \in M_j$.

Since bipartite matching is a classic algorithm with complexity $O(n^2)$, Algorithm 2 is polynomial.

Call two graphs G_A and G_B *equispectral* if the set (not multiset) of non-zero eigenvalues of G_A coincides with the set of non-zero eigenvalues of G_B. According to the following theorem, any pair of non-equispectral graphs are distinguished by equivalence.

Theorem 3.2. *If $G_A \equiv G_B$, then G_A and G_B are equispectral.*

Proof. Assume G_A and G_B are not equispectral, and let $\lambda \neq 0$ be an eigenvalue of G_A but not of G_B. Let E_λ be the eigenspace of G_A corresponding to λ and let $\mathbf{e}$ be

a standard vector with non-zero orthogonal projection on E_λ. By way of contradiction, assume that $G_A \equiv G_B$. Then there is a permutation matrix P such that $A^t\mathbf{e} = (P^{-1}BP)^t\mathbf{e}$ for $t \geq 0$. Let $B' = P^{-1}BP$ and denote by λ_i and μ_i the distinct eigenvalues of A and B', respectively, with $\lambda = \lambda_1$. Then $\sum_i \mathbf{x}_i = \mathbf{e} = \sum_i \mathbf{x}'_i$, where $\mathbf{x}_i$ and $\mathbf{x}'_i$ are the projections of $\mathbf{e}$ on the eigenspaces E_{λ_i} and E_{μ_i}, respectively. Hence

$$\sum_i \lambda_i^t \mathbf{x}_i = A^t\mathbf{e} = B'^t\mathbf{e} = \sum_i \mu_i^t \mathbf{x}'_i, \qquad t = 0, 1, 2, \ldots. \tag{3.1}$$

It may be that some λ_i equal some μ_j. In any case, terms with the same eigenvalues in Eq. (3.1) may be collected to obtain

$$\lambda^t \mathbf{x}_1 + \sum_{i=2}^{s} \eta_i^t \mathbf{y}_i = \mathbf{0} \tag{3.2}$$

for some distinct non-zero real numbers η_i and vectors $\mathbf{y}_i$. Some coordinate of $\mathbf{x}_1$, say the kth coordinate, is non-zero. Considering only the kth coordinate in Eq. (3.2) yields the linear system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda & \eta_2 & \cdots & \eta_s \\ \lambda^2 & \eta_2^2 & \cdots & \eta_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{s-1} & \eta_2^{s-1} & \cdots & \eta_s^{s-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $a_1 \neq 0$. But this Vandermonde matrix is non-singular, which is a contradiction. □

Next extend the notion of equivalence as follows. Let m be a natural number and let $\binom{[n]}{m}$ denote the collection of m-element subsets of $\{1, 2, \ldots, n\}$. For each $I = (i_1, i_2, \ldots, i_m) \in \binom{[n]}{m}$, with elements in a particular order, let

$$\mathbf{A}_I := \mathbf{A}_{(i_1, i_2, \ldots, i_m)}$$

denote the concatenation of the graph recurrence sequences $\mathbf{A}_{i_1}, \mathbf{A}_{i_2}, \ldots, \mathbf{A}_{i_m}$. A pair of graphs G_A and G_B will be called *m-equivalent*, denoted

$$G_A \equiv_m G_B,$$

if there exists a bijection $f : \binom{[n]}{m} \to \binom{[n]}{m}$ such that

$$\mathbf{A}_I \equiv \mathbf{B}_{f(I)}$$

for some ordering of the elements of I and $f(I)$. Intuitively, m-equivalence means that there is a bijection between m-element subsets of vertices of the two graphs such that for each corresponding pair of m-element subsets (in some order), all m pairs of corresponding graph recurrence sequences are identical (*using a single permutation*).

Note that 1-equivalence is the same as equivalence. The following extension of Algorithm 2 shows that m-equivalence is also testable in polynomial time.

Algorithm 3.

Input Graphs G_A and G_B.

Output Whether or not $G_A \equiv_m G_B$.

1. For each $I \in \binom{[n]}{m}$, let $M_I = \{J \in \binom{[n]}{m} \mid \mathbf{A}_I \equiv \mathbf{B}_J\}$. The set M_I is determined by $m!\binom{[n]}{m}$ repetitions of Algorithm 1, a polynomial in n since m is a fixed constant. (The $m!$ comes from the possible necessity of testing each permutation of the elements of J.)
2. A bijection $f : \binom{[n]}{m} \to \binom{[n]}{m}$ such that $\mathbf{A}_I \equiv \mathbf{B}_{f(I)}$ for every I is given by a system of distinct representatives of $\{M_I \mid I \in \binom{[n]}{m}\}$. Exactly as in Algorithm 2, this is equivalent to finding a perfect matching in a bipartite graph.

Remark 3.3. A concept stronger than m-equivalence would be obtained by requiring f to be a bijection from $[n]$ onto $[n]$ rather than from $\binom{[n]}{m}$ onto $\binom{[n]}{m}$. However, for this more stringent definition, there is probably no polynomial time test for m-equivalence.

Proposition 3.4. *If $G_A \equiv_m G_B$, then $G_A \equiv_k G_B$ for all $k < m$.*

Proof. Assume that $G_A \equiv_m G_B$, $m > 1$. To be precise, we are given a bijection $f : \binom{[n]}{m} \to \binom{[n]}{m}$ and a permutation $\alpha_I : I \to I$ for each $I \in \binom{[n]}{m}$ such that $\mathbf{A}_I \equiv \mathbf{B}_{\alpha_I(f(I))}$.

It is sufficient to show that $G_A \equiv_{m-1} G_B$. And for this it suffices to prove the existence of a bijection $f' : \binom{[n]}{m-1} \to \binom{[n]}{m-1}$, where, for each $I' \in \binom{[n]}{m-1}$, there is an $I \in \binom{[n]}{m}$ such that $I' \subset I$ and $\alpha'_{I'}$ is the restriction of α_I to I'.

For a given $I' \in \binom{[n]}{m-1}$ let

$$A_{I'} = \left\{\alpha_I(I') \mid I' \subset I \in \binom{[n]}{m}\right\}$$

be the set of possible $(m-1)$-element subsets to which I' can potentially be mapped by f'. A map f' is equivalent to a system of distinct representatives of the collection of sets $\{A_{I'} \mid I' \in \binom{[n]}{m-1}\}$.By way of contradiction, assume that no such system of distinct representatives exists. By P. Hall's theorem on distinct representatives there exists an $S \subset \binom{[n]}{m-1}$ such that $|T| < |S|$, where

$$T = \bigcup\{A_{I'} \mid I' \in S\}.$$

Now

$$\begin{aligned} |S|(n-m+1) &= \left|\left\{(I', I) \mid I' \in S, I' \subset I \in \binom{[n]}{m}\right\}\right| \\ &= \left|\left\{(\alpha_I(I'),\ f(I)) \mid I' \in S, I' \subset I \in \binom{[n]}{m}\right\}\right| \\ &< \left|\left\{I', I) \mid I' \in T, I' \subset I \in \binom{[n]}{m}\right\}\right| = |T|(n-m+1). \end{aligned}$$

This implies that $|T| > |S|$, a contradiction. □

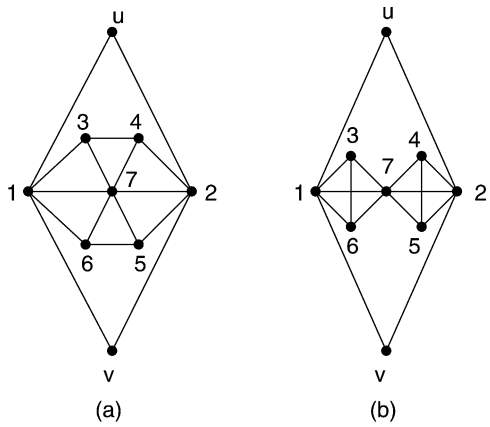

Fig. 2. Construction of non-isomorphic, 2-equivalent graphs.

Example 3.5 (Non-isomorphic, 2-equivalent graphs). The construction of such a pair of graphs is as follows. Let H_1 and H_2 be two non-isomorphic graphs on n vertices with the same degree sequence. Such pairs are well known to exist, for example a pair of non-isomorphic regular graphs of the same degree. Consider two copies of the complete graph K_n. In the first copy, label the edges of a subgraph isomorphic to H_1 by (a) and the remaining edges (b). Replace each edge $\{u, v\}$ labeled (a) by the graph in Fig. 2(a) and each edge $\{u, v\}$ labeled (b) by the graph in Fig. 2(b). Call the resulting graph G_1. For the second copy of K_n do the same thing with respect to H_2. Call the resulting graph G_2.

Theorem 3.6. *The graphs G_1 and G_2 described above are 2-equivalent but are not isomorphic.*

Proof. Although the graphs G_1 and G_2 have the same order, it is clear that they are not isomorphic. Consider either of the two graphs in Fig. 2. Partition the vertices as follows: $(u)(v)(1\,2)(3\,4\,5\,6)(7)$. Notice that, for any initial vector that is constant on each block of this partition, each term in the graph recurrence sequence is also constant on each block and, in fact, has the same value whether it is the graph of Fig. 2(a) or 2(b).

A subgraph of G_1 or G_2 of the type in Fig. 2(a) will be referred to as a subgraph of type (a); similarly a subgraph of the type in Fig. 2(b) will be referred to as a subgraph of type (b). Let w be any vertex of G_1. Consider any bijection of the vertices of K_n onto the vertices of K_n and extend to a bijection ϕ between the vertices of G_1 and G_2 that preserves labels $\{u, v, 1, 2, 3, 4, 5, 6, 7\}$. Moreover, if w is not a vertex labeled u or v in Fig. 2, then ϕ should be chosen so that if w lies in a subgraph of type (a) (type (b)), then $\phi(w)$ also lies in a subgraph of type (a) (type (b)). Consider the initial vertex labeling of G_1 with value 1 at vertex w and value 0 at all vertices, and the initial vertex labeling of G_2 with value 1 at vertex $\phi(w)$ and value 0 at all vertices. Then, by the comments in the paragraph above, it is easy to prove by induction that the corresponding graph recurrence sequences for G_1 and G_2 are identical.

To emphasize the role of the adjacency matrices, denote the two graphs by G_A and G_B instead of G_1 and G_2. To show that $G_A \equiv_2 G_B$ construct a bijection $f : \binom{[N]}{2} \to \binom{[N]}{2}$ such that $\mathbf{A}_I \equiv \mathbf{B}_{f(I)}$ as follows. There are two cases.

Case 1. Consider a pair (w, z) of vertices in G_A, both in the same subgraph of type (a). This pair corresponds under f to a pair (w', z') of vertices from any subgraph of type (a) in G_B. The vertices w, w' should have the same labels from $\{u, v, 1, 2, 3, 4, 5, 6, 7\}$; similarly for the vertices z, z'. It is possible to define such a bijection because G_A and G_B have the same number of subgraphs of type (a). This is because the graphs H_1 and H_2 have the same degree sequence and hence the same number of edges. The bijection f is defined analogously for a pair of vertices in G_A, both in a subgraph of the type (b), because the complements of H_1 and H_2 have the same number of edges. Clearly there is a bijection ϕ from the set of vertices of G_A onto the set of vertices of G_B as described in the second paragraph of this proof so that $(\phi(w), \phi(z)) = (w', z') = f(w, z)$. Then $\mathbf{A}_I \equiv \mathbf{B}_{f(I)}$ for all such pairs $I = (w, z)$.

Case 2. Because H_1 and H_2 have the same degree sequence, the two copies of K_n have the same number of pairs of incident edges both labeled (a), and hence the same number of non-incident pairs of edges both labeled (a). Let F_1 and F_2 be the subgraphs of the two copies of K_n induced by the edges labeled (b). Then F_1 and F_2 also have the same degree sequence; hence the two copies of K_n have the same number of pairs of incident edges both labeled (b) and the same number of pairs of non-incident edges both labeled (b). That H_1 and H_2 have the same degree sequence also implies that the number of pairs of incident edges, one labeled (a) the other (b), is the same in both copies of K_n, which, in turn, implies the same for non-incident edges. Now consider pairs of vertices in G_A that are contained in distinct subgraphs of type (a) (respectively type (b)) that come from incident edges (respectively non-incident edges) in K_n. Then, by the comments above, these pairs of vertices can be bijectively matched with pairs of vertices in G_B that are contained in distinct subgraphs of type (a) (respectively type (b)) that come from incident edges (respectively non-incident edges) in K_n. This bijection f should be such that, for corresponding pairs (w, z) and (w', z'), the vertices w, w' have the same labels from $\{u, v, 1, 2, 3, 4, 5, 6, 7\}$; similarly for z, z'. Again there is clearly a bijection ϕ from the set of vertices of G_A onto the set of vertices of G_B as described in the second paragraph of this proof so that $(\phi(w), \phi(z)) = (w', z') = f(w, z)$. Then $\mathbf{A}_I \equiv \mathbf{B}_{f(I)}$ for all such pairs $I = (w, z)$. □

The graphs in the example are 2-connected, but not 3-connected. A 3-connected example can be obtained by joining in G_1 (and also G_2) each vertex labeled 7 to each vertex of the original K_n.

Questions 3.7. The first question below is probably difficult since an affirmative answer would imply a polynomial time algorithm for the graph isomorphism problem.

1. Does there exist a fixed integer m such that $G_1 \equiv_m G_2$ implies $G_1 \approx G_2$.
2. What is the least m such that $G_1 \equiv_m G_2$ implies $G_1 \approx G_2$ for connected planar graphs?

4. M-regular graphs

From the point of view of the graph isomorphism problem, strongly regular graphs have proved particularly troublesome. Several known algorithms that are fast for arbitrarily selected graphs fail badly for non-isomorphic pairs of strongly regular graphs with the same parameters. In fact, there are random graph algorithms that run in polynomial time for almost all graphs and fail for strongly regular graphs [1].

This section concerns a generalization of strongly regular graphs and indicates why such graphs are problematic. For a given induced subgraph H of a graph G, let $\lambda(H)$ be the number of vertices of $G \setminus H$ adjacent to all vertices of H. For a non-negative integer m call a graph G *m-regular* if $\lambda(H)$ depends only on the isomorphic type of H for all induced subgraphs H with $\text{order}(H) \leq m$. Call the set of values $\{\lambda(H) \mid \text{order}(H) \leq m\}$ the *parameter set* of G. Vacuously, all graphs are 0-regular with parameter set $\{n\}$, the order of the graph. A graph is 1-regular if and only if it is regular in the ordinary sense, i.e., each vertex has the same degree k. The parameter set is $\{n, k\}$.

Example 4.1 (Strongly regular graphs). A graph is 2-regular if and only if it is strongly regular. Recall that a graph is *strongly regular* if it is regular of degree k with constants λ and μ such that

1. any pair of adjacent vertices are mutually adjacent to exactly λ vertices, and
2. any pair of non-adjacent vertices are mutually adjacent to exactly μ vertices.

Strongly regular graphs arise from certain 2-designs and are also combinatorial generalizations of graphs naturally associated with rank 3 permutation groups. The parameter set of a 2-regular graph is $\{n, k, \lambda, \mu\}$. The Petersen graph, for example, is strongly regular with parameter set $\{10, 3, 0, 1\}$.

Example 4.2. Consider the graph G whose vertex set is the Cartesian product $[N] \times [N]$, where two vertices (a, b) and (a', b') are adjacent in G if $a \neq a'$ and $b \neq b'$. It can be shown that G is 3-regular. If $N = 3$ then G is the line graph of K_{33}, which happens to be m-regular for all m, as is the case for the pentagon.

Example 4.3. Complete graphs and complete and multipartite graphs are m-regular for all m. This is clear because, for any pair H, H' of isomorphic induced subgraphs, there is an automorphism of G taking H onto H'.

Theorem 4.4. *If G and G' are m-regular graphs, $m \geq 2$, with the same parameter set, then G and G' are m-equivalent.*

The proof of Theorem 4.4 requires the following two lemmas. Let H be an induced subgraph of G and J an induced subgraph of H. The pairs (H, J) and (H', J') will be called *isomorphic* if there exists an isomorphism of H onto H' taking J onto J'. Let $\Gamma(G, H, J)$ denote the set of vertices of $G \setminus H$ adjacent to each vertex in J and non-adjacent to each vertex of $H \setminus J$, and let $\lambda(G, H, J) = |\Gamma(G, H, J)|$.

Lemma 4.5. *Given an m-regular graph G, let H be an induced subgraph of G with $order(H) \leq m$ and J an induced subgraph of H. Then $\lambda(G, H, J)$ depends only on the isomorphism type of (H, J).*

Proof. To simplify notation, we make no distinction between a set S of vertices and the subgraph of G that these vertices induce. For a subset S of vertices of H, let $\alpha(H, S)$ denote the number of vertices in $H - S$ adjacent to all vertices of S. The lemma then follows from the Inclusion–Exclusion Principle:

$$\lambda(G, H, J) = \sum_{J \subseteq S \subseteq H} (-1)^{|S|-|J|} [\lambda(S) - \alpha(H, S)]$$

because $\lambda(S)$ depends only the isomorphism type of S, $\alpha(H, S)$ depends only on S, and the possibilities for S depend only on the isomorphism type of (H, J). □

Lemma 4.6. *If G_1 and G_2 are m-regular with the same set of parameters then, for any graph H of order $\leq m$, G_1 and G_2 contain the same number of induced isomorphic copies of H.*

Proof. The proof is by induction on the order H. If the order is 1, then H is a single vertex. But both graphs are 0-regular with parameter set $\{n\}$, where n is the number of vertices. Next assume that the lemma is true for order $k - 1 < m$ and let H have order k. Let v be a vertex of H and let H' be the graph obtained from H by deleting v. In fact, there may be several vertices, say r of them, each of whose removal from H results in H' up to isomorphism. Given H', let J be an induced subgraph of H' such that, if a new vertex v, not in H', is joined to each vertex of J, the resulting graph is isomorphic to H. In fact, there may be several such subgraphs, say $J_1, J_2, \ldots, J_s$. Let $c(G_j, H')$ denote the number of induced copies of H' in G_j, $j = 1, 2$. Then the number of induced copies of H in G_j is

$$c(G_j, H) = \frac{1}{r} \sum_{i=1}^{s} c(G_j, H')\lambda(G_j, H, J_i).$$

By the induction hypothesis $c(G_1, H') = c(G_2, H')$, and by Lemma 4.5 $\lambda(G_1, H, J_i) = \lambda(G_2, H, J_i)$ for each i. Therefore $c(G_1, H) = c(G_2, H)$. □

Proof of Theorem 4.4. Assume that G and G' are m-regular, $m \geq 2$, with the same set of parameters. Label the vertices of each graph $\{1, 2, \ldots, n\}$. By the previous lemma, there exists a bijection between the set of all induced subgraphs of G of order m and the set of all induced subgraphs of G' of order m such that corresponding subgraphs are isomorphic. This provides a bijection $f : \binom{[n]}{m} \to \binom{[n]}{m}$ (with the ordering on corresponding m-sets provided by the isomorphism).

For a given $I \in \binom{[n]}{m}$, let H be the corresponding subgraph of G. Similarly let H' be the subgraph of G' corresponding to $f(I)$. Define a bijection $\phi : V(G) \to V(G')$ as follows. The mapping ϕ restricted to H is an isomorphism from H to H'. Note that

$$\{\Gamma(G, H, J) \mid J \text{ an induced subgraph of } H\} \tag{4.1}$$

forms a partition of $V(G \setminus H)$. For any induced subgraph J of H, let $J' = \phi(J)$. Using this partition and Lemma 4.5, the function ϕ can be extended to a bijection from $V(G)$ to $V(G')$ such that $\phi : \Gamma(G, H, J) \to \Gamma(G', H', J')$.

Denote the value of the jth coordinate of a vector $\mathbf{x}$ by $\mathbf{x}(j)$. Let v_0 be any vertex of H. In G let

$$\mathbf{x}_0(v) = \begin{cases} 1 & \text{if } v = v_0 \\ 0 & \text{otherwise} \end{cases}$$

and in G' let

$$\mathbf{x}'_0(v) = \begin{cases} 1 & \text{if } v = \phi(v_0) \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the graph recurrence sequences. It suffices to show that $\mathbf{x}'_t(\phi v) = \mathbf{x}_t(v)$ for all $v \in V(G)$ and all t. To prove this let $\Gamma(v)$ and $\Delta(v)$ denote the set of neighbors and non-neighbors of vertex v, respectively. To simplify notation denote $\lambda(G, H, J)$ simply by $\lambda(H, J)$. Also $\bullet - \bullet$ and $\bullet \quad \bullet$ denote the two possible graphs on two vertices. We show by induction that the value of $\mathbf{x}_t$, $t = 0, 1, \ldots$ is constant on $\Gamma(v_0)$ and $\Delta(v_0)$. Denote these values by $\mathbf{x}_t(\Gamma)$ and $\mathbf{x}_t(\Delta)$, respectively. Using the 2-regularity of G, the induction is as follows:

$$\mathbf{x}_0(v) = \begin{cases} 1 & \text{if } v = v_0 \\ 0 & \text{if } v \in \Gamma(v_0) \\ 0 & \text{if } v \in \Delta(v_0) \end{cases}$$

and

$$\mathbf{x}_{t+1}(v) = \begin{cases} \lambda(\bullet, \bullet)\mathbf{x}_t(\Gamma) & \text{if } v = v_0 \\ \mathbf{x}_t(v_0) + \lambda(\bullet - \bullet, \bullet - \bullet)\mathbf{x}_t(\Gamma) + \lambda(\bullet - \bullet, \bullet)\mathbf{x}_t(\Delta) & \text{if } v \in \Gamma(v_0) \\ \lambda(\bullet \quad \bullet, \bullet \quad \bullet)\mathbf{x}_t(\Gamma) + \lambda(\bullet \quad \bullet, \bullet)\mathbf{x}_t(\Delta) & \text{if } v \in \Delta(v_0). \end{cases}$$

Exactly the same equations are true for $\mathbf{x}'_t(v)$ with v_0 replaced by ϕv_0. Since G and G' are 2-regular with the same parameters, the corresponding λ's in the two sets of equations are the same.

Note that $\Gamma(v_0) \cap (G \setminus H)$ and $\Delta(v_0) \cap (G \setminus H)$ are unions of sets in the partition Eq. (4.1). Also $\phi(\Gamma(v_0) \cap H) = \Gamma(\phi(v_0)) \cap H'$ and $\phi(\Delta(v_0) \cap H) = \Delta(\phi(v_0)) \cap H'$. Therefore, by its definition, ϕ takes $\Gamma(v_0)$ onto $\Gamma(\phi v_0)$ and $\Delta(v_0)$ onto $\Delta(\phi v_0)$. Hence $\mathbf{x}'_t(\phi v) = \mathbf{x}_t(v)$ for all $v \in V(G)$ and all t. □

If there exists a pair of non-isomorphic m-regular graphs with the same set of parameters then, according to Theorem 4.4, this pair also shows that m-equivalence is invalid as a polynomial test for graph isomorphism. That was our motivation in considering m-regularity. The follow result, however, offers some hope.

Theorem 4.7. *For $m \geq 5$ there does not exist a pair of non-isomorphic m-regular graphs with the same parameters.*

Proof. The proof is essentially due to Cameron, Goethals and Seidel as described in [3]. They show that a 5-regular graph is one of the following: a disjoint union of complete graphs, a complete multipartite graph, a pentagon, or the line graph of $K_{3,3}$. □

Questions 4.8. The following question has implications for the answer to the first of Questions 3.7.

1. Does there exist a pair of 3-regular graphs with the same set of parameters?

5. Separating vertices

A set Y of vectors in $\mathbb{R}^n$ is said to *separate* coordinates $i, j \in [n]$ if $y(i) \neq y(j)$ for some $y \in Y$. If, in a graph G_A, the set of terms $\{A^t\mathbf{x} \mid t \geq 0, \mathbf{x} \in X\}$ in the graph recurrence sequences with initial values in X separates coordinates i and j, then we say that X *separates vertices* i and j. And if $X = \{\mathbf{e}_i \mid i \in I\}$ is a set of standard vectors, then we simply say that vertex subset I *separates* i and j. If every pair of vertices of G is separated by the set I of vertices, we say that I *separates* G. In Fig. 1, for example, vertex 1 separates vertices 2 and 4 but does not separate vertices 2 and 3.

Note that no single vertex separates the complete graph K_n. In fact, the graph recurrence sequence is: $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 1)$, $(n-1, n-2, \ldots, n-2), \ldots$, so that no pair among the last $n-1$ vertices is separated. On the other hand, Theorem 2.6 guarantees that a single vertex determines K_n. So it is possible that a vertex determines a graph but does not separate it. The following question concerns the converse.

Question 5.1. If I separates a graph G, does I determine G?

Theorem 5.2. *Let $\mathcal{G}_m$ denote the class of graphs G for which there is some set consisting of at most m vertices that separates G. For the class $\mathcal{G}_m$ the graph isomorphism problem is polynomial.*

Using the minimal polynomial of the adjacency matrix as in the proof of Lemma 2.1, it can be shown that two vertices of a graph on n vertices are separated by the graph recurrence sequences with initial vectors in some set X if and only if the two vertices are already separated by the first n terms of the graph recurrence sequences with initial vectors in X. Therefore, for a fixed m, determining whether or not a graph is separated by a set of m vertices is computationally polynomial. The test for isomorphism, and thus the proof of Theorem 5.2, is as follows. We use the notation of Section 3.

Algorithm 4.

Input Two graphs G_A and G_B on n vertices in the class $\mathcal{G}_m$.

Output Whether or not $G_A \approx G_B$.

For each m-subset $I \subset [n]$ determine whether I separates G_A. For each such I that separates G_A, perform the following steps:

1. Use Algorithm 1 to find $M_I = \{J \in \binom{[n]}{m} \mid \mathbf{A}_I \equiv \mathbf{B}_J\}$.

2. For each $J \in M_I$ use Algorithm 1 to find the unique bijection $g : V(G_A) \to V(G_B)$ such that $\mathbf{A}_I \equiv \mathbf{B}_J$.
3. Check each such bijection g to determine whether it is indeed an isomorphism between G_A and G_B. If such a g exists then $G_A \approx G_B$; otherwise it is not the case that $G_A \approx G_B$.

Example 5.3 (Graphs not in $\mathcal{G}_m$ for any m). This example shows that the above algorithm will not extend from $\mathcal{G}_m$ to the collection of all graphs. Fix an arbitrary positive integer m. The complete and complete bipartite graphs with sufficiently many vertices do not lie in $\mathcal{G}_m$. For a more convincing example consider the strongly regular graphs defined in Section 4. For a vertex i in a strongly regular graph G_A on n vertices, let $\Gamma(i)$ and $\Delta(i)$ denote the set of neighbors and non-neighbors of vertex i, respectively. It follows from the definition of strongly regular that $A^t\mathbf{e}_i$ is constant on $\Gamma(i)$ and constant on $\Delta(i)$, for all $t \geq 0$. This implies that at least $\log_2 n - 1$ vertices are required to separate G_A.

We next find an upper bound on how finely a set $I \subset [n]$ can separate the vertices of a graph G_A. For a graph G_A and subset I of vertices, we will define three related partitions, based on graph recurrence sequences, on the automorphism group of G and on the centralizer algebra of the automorphism group, respectively:

$$\pi_A(I) \qquad \pi_\Gamma(I) \qquad \pi_C(I).$$

For a subset $I \subset [n]$ of vertices, consider the equivalence relation $i \sim j$ if i and j are not separated by I. Let $\pi_A(I)$ denote the resulting partition of $[n]$.

Let $\Gamma := \Gamma(G)$ denote the automorphism group of G and Γ_i the stabilizer subgroup of vertex i. For a subset $I \subset [n]$ let $\pi_\Gamma(I)$ be the partition of $[n]$ such that two elements belong to the same block of $\pi_\Gamma(I)$ if and only if they are in the same orbit of each subgroup Γ_i, $i \in I$.

Representing the elements of the automorphism group Γ as permutation matrices, the *centralizer algebra* of the automorphism group is defined by

$$C := C(\Gamma) = \{B \in Mat_{n\times n}(\mathbb{C}) \mid BP = PB \text{ for all } P \in \Gamma\}.$$

For a subset $I = \{i_1, \ldots, i_m\}$ let

$$C(I) = \{B\mathbf{e}_i \mid B \in C(\Gamma), i \in I\}.$$

Consider the equivalence relation defined by $i \sim j$ if i and j are not separated by $C(I)$. Let $\pi_C(I)$ denote the resulting partition of $[n]$.

For the graph G_A in Fig. 1 take I to consist of the single vertex 1. In this case $\pi_\Gamma(I) = \pi_C(I) = \pi_A(I) = (1)(23)(4)$. In the next theorem $\pi \leq \sigma$ means that partition π is finer than partition σ, i.e., every block of π is a block of σ.

Theorem 5.4. *If G is a graph on n vertices and $I \subseteq [n]$, then*

$$\pi_\Gamma(I) = \pi_C(I) \leq \pi_A(I).$$

Proof. The $\leq$ is due to the fact that the adjacency matrix of a graph, and all of its powers, are members of the centralizer algebra of the automorphism group.

Concerning the equality, it is first shown that $\pi_\Gamma(I) \leq \pi_C(I)$. Assume that j and k are in the same block of $\pi_\Gamma(I)$. Then for each $i \in I$ there is an automorphism $g := g_i \in \Gamma_i(G)$ such that $g(j) = k$. By abuse of language also let g denote the corresponding permutation matrix. Now if $B \in C(\Gamma)$ and $\mathbf{e} := \mathbf{e}_i$, then $(B\mathbf{e})(j) = [(g^{-1}Bg)\mathbf{e}](j) = g^{-1}[(Bg)(\mathbf{e})](j) = [(Bg)(\mathbf{e})](k) = [B(g\mathbf{e})](k) = (B\mathbf{e})(k)$. Hence j and k are in the same block of $\pi_C(I)$.

To show that $\pi_C(I) \leq \pi_\Gamma(I)$, assume that j and k are in distinct blocks of $\pi_\Gamma(I)$. We will show that j and k are in distinct blocks of $\pi_C(I)$. Since j and k are in distinct blocks of $\pi_\Gamma(I)$, then for each $i \in I$ there is no automorphism of G that fixes i and takes j to k.

Consider the action of the group $\Gamma(G)$ on $[n] \times [n]$ defined by $g(s,t) = (gs, gt)$. Let O be any orbit under this action and define a matrix $B_O = (b_{s,t})$ by $b_{s,t} = 1$ if $(s,t) \in O$ and $b_{s,t} = 0$ otherwise. It is straightforward to check that B_O is in the centralizer $C(\Gamma)$. (In fact the set of such matrices forms a basis for the centralizer.) Consider the orbit O of the pair (j,i) under this action. Because (k,i) is not in O, $B_O\mathbf{e}_i$ distinguishes j and k. □

According to Theorem 5.4 the graph recurrence sequence cannot separate points any finer than the automorphism group or centralizer algebra. The inequality in the theorem is, in general, strict. As an example, let π denote a partition of the set $[n]$ into blocks $V_1, V_2, \ldots, V_s$. Let $D = (d_{ij})$ be an $s \times s$ matrix. Call G a *graph of type* (π, D) if π is a partition of the vertex set into blocks $V_1, V_2, \ldots, V_s$ and each vertex of block V_i is joined to exactly d_{ij} vertices of block V_j. Any regular graph of degree d, for example, is a graph of type (π, D) where $\pi = (1, 2, \ldots, n)$ is the partition with just one block and $D = (d)$. At the other extreme, if $\pi = (1)(2)\ldots(n)$ is the trivial partition into blocks all of cardinality 1, then any graph is of type (π, A) where A is the adjacency matrix. In general, there can be many graphs of a given type, many with trivial symmetry group. Consider such an asymmetric graph G were V_1 consists of a single vertex, say vertex 1. Then clearly the stabilizer of 1 separates the vertices of G whereas the graph recurrence sequence centered at vertex 1 does not.

Questions 5.5. If $\pi_\Gamma(I) = (1)(2)\cdots(n)$, then we say that I *separates the vertices of* G *with respect to the automorphism group*. Call the smallest cardinality of such a separating set I the *separation index of* G *with respect to its automorphism group*, denoted $\mathrm{sep}_\Gamma(G)$. Similarly if $\pi_A(I) = (1)(2)\cdots(n)$ we say that I *separates the vertices of* G *with respect to graph recurrence sequences*. Call the smallest cardinality of such a separating set the *separation index of* G *with respect to its graph recurrence sequences*, denoted $\mathrm{sep}_A(G)$. For a collection $\mathcal{G}$ of graphs let

$$\mathrm{sep}_\Gamma(\mathcal{G}) = \min\{\mathrm{sep}_\Gamma(G) \mid G \in \mathcal{G}\}$$
$$\mathrm{sep}_A(\mathcal{G}) = \min\{\mathrm{sep}_A(G) \mid G \in \mathcal{G}\}.$$

1. Let $\mathcal{G}_m$ denote the collection of graphs G of degree at most m such that both G and its complement are connected. Find $\mathrm{sep}_\Gamma(\mathcal{G}_m)$ and $\mathrm{sep}_A(\mathcal{G}_m)$. Because any automorphism of G induces an automorphism of its complement, the condition that G and its complement be connected is necessary. If $\mathrm{sep}_A(\mathcal{G}_m)$ exists, then Algorithm 4 is a polynomial procedure using graph recurrence sequences to solve the graph isomorphism problem for graphs of bounded degree.

2. Let $\mathcal{P}$ denote the collection of connected planar graphs G such that both G and its complement are connected. Is it true that $\text{sep}_A(\mathcal{P}) = 3$? Note that $\text{sep}_A(\mathcal{P}) \neq 2$ because the graph of the 3-cube cannot be separated by any two vertices. Also note that 3 does not suffice for the families of planar graphs $K_1 \vee \overline{K_n}$ and $\overline{K_2} \vee \overline{K_n}$, whose complements are not connected. Here $\vee$ denotes the disjoint union, each vertex of one graph adjacent to each vertex of the other. In [9] we prove that $\text{sep}_\Gamma(\mathcal{P}) = 3$ for any 3-connected planar graph.

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