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On the structure of k-connected graphs without K_k-minor

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Abstract

Suppose G is a k-connected graph that does not contain K_k as a minor. What does G look like? This question is motivated by Hadwiger's conjecture (Vierteljahrsschr. Naturforsch. Ges. Zürich 88 (1943) 133) and a deep result of Robertson and Seymour (J. Combin. Theory Ser. B. 89 (2003) 43).

It is easy to see that such a graph cannot contain a $(k-1)$-clique, but could contain a $(k-2)$-clique, as $K_{k-5}+G'$, where G' is a 5-connected planar graph, shows. In this paper, however, we will prove that such a graph cannot contain three "nearly" disjoint $(k-2)$-cliques. This theorem generalizes some early results by Robertson et al. (Combinatorica 13 (1993) 279) and Kawarabayashi and Toft (Combinatorica (in press)).

1. Introduction and notation

Hadwiger's conjecture from 1943 suggests a far-reaching generalization of the Four Color Problem, and it is perhaps the most interesting conjecture in graph theory. Hadwiger's conjecture states the following.

Conjecture 1.1 ([5]). *For all $k \geq 1$, every k-chromatic graph has the complete graph K_k on k vertices as a minor.*

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For $k = 1, 2, 3$, it is easy to prove, and for $k = 4$, Hadwiger himself [5] and Dirac [4] proved it. For $k = 5$, however, it seems extremely difficult. In 1937, Wagner [22] proved that the case $k = 5$ is equivalent to the Four Color theorem. So, assuming the Four Color theorem [1, 2, 14], the case $k = 5$ in Hadwiger's conjecture holds. In 1993, Robertson, Seymour and Thomas [17] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. Hence, assuming the Four Color theorem, the case $k = 6$ of Hadwiger's conjecture holds. This result is the deepest in this research area. So far, the cases $k \geq 7$ are open.

Motivated by Hadwiger's conjecture, the following question is drawn attention to by many researchers.

Question 1.2. *What do K_k-minor-free graphs look like?*

One approach is to consider the maximal size of graphs not having K_k as a minor. Wagner [23] showed that a sufficiently large chromatic number (which depends only on k) guarantees a K_k as a minor, and Mader [11] showed that a sufficiently large average degree will do. Kostochka [10], and Thomason [19], independently, proved that $k\sqrt{\log k}$ is the correct order for the average degree because random graphs having no K_k-minor may have average degree of order $k\sqrt{\log k}$. (Recently, Thomason [20] gave a more exact "extremal" function.)

Another approach is due to Robertson and Seymour [15]. They considered how to construct graphs with no K_k-minor. If G contains a set X with at most $k - 5$ vertices such that $G - X$ is planar, G does not contain K_k as a minor since planar graphs cannot contain K_5 as a minor. Similarly, if G contains a set X with at most $k - 7$ vertices such that $G - X$ can be drawn in the projective plane, then clearly G does not contain K_k as a minor. (Since the projective plane cannot contain K_7 as a minor.) Or if G contains a set with at most $k - 8$ vertices such that $G - X$ can be drawn in the torus, then clearly G does not contain K_k as a minor. (Again, the torus cannot contain K_8 as a minor.) These observations together with the concept "clique-sum" led Robertson and Seymour to one of their celebrated results of excluding the complete graph minor, and this is the most important step in their proof of "Wagner's conjecture" [16].

Our motivation is the following question.

Question 1.3. *What do K_k-minor-free k-connected graphs look like?*

It does not seem that random graphs give an answer to this question because, as Thomason [20] pointed out, extremal graphs are more or less exactly vertex disjoint unions of suitable dense random graphs. It does not seem that Robertson and Seymour's excluded minor theorem gives an answer either, because their characterization does not seem to guarantee high connectivity. In view of these observations, we still do not know what K_k-minor-free k-connected graphs look like.

The following question is also motivated by Hadwiger's conjecture.

Question 1.4. *Is it true that a minimal counterexample to Hadwiger's conjecture for $k \geq 6$ has a set X of $k - 5$ vertices such that $G - X$ is planar?*

This is true for $k = 6$ as Robertson et al. [17] showed. To consider a minimal counterexample to Hadwiger's conjecture, one can prove the following conjecture.

Conjecture 1.5. *A minimal counterexample to Hadwiger's conjecture is k-connected.*

This is true for $k \leq 7$ as Mader proved in [12]. Note that Toft [21] proved that a minimal counterexample to Hadwiger's conjecture is k-edge-connected. This gives a strong evidence to Conjecture 1.5.

Question 1.4 and Conjecture 1.5 lead us to the following question.

Question 1.6. *Is it true that a K_k-minor-free k-connected graph for $k \geq 6$ has a set X of $k-5$ vertices such that $G - X$ is planar?*

The case $k = 6$ is a well-known conjecture due to Jørgensen [7], and still open. If true, this would imply Hadwiger's conjecture for $k = 6$ case by Mader's result [11]. The case $k = 7$ was conjectured in [8] as well.

Even though the case $k = 6$ of the Question 1.6 is still open, Robertson et al. [17] gave a result for searching K_6-minor.

Theorem 1.7 ([17]). *Let G be a simple 6-connected non-apex graph. If G contains three 4-cliques, say, L_1, L_2, L_3, such that $|L_i \cap L_j| \leq 2$ $(1 \leq i < j \leq 3)$, then G contains a K_6 as a minor.*

Recently, Kawarabayashi and Toft [8] proved the following theorem.

Theorem 1.8. *Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor.*

This settles the case $(6, 1)$ of the following conjecture known as the $(k-1, 1)$-minor conjecture which is a relaxed version of Hadwiger's conjecture:

Conjecture 1.9 ([3, 24]). *For all $k \geq 1$, every k-chromatic graph has either a K_k-minor or a $K_{\lfloor \frac{k+1}{2} \rfloor, \lceil \frac{k+1}{2} \rceil}$-minor.*

In [8], the following result is the key lemma, and gave a result for searching K_7-minor.

Theorem 1.10 ([8]). *Let G be a 7-connected graph with at least 19 vertices. Suppose G contains three 5-cliques, say, L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 12$, then G contains a K_7-minor.*

Our work is motivated by Theorems 1.7 and 1.10, and the main result of this paper is the following theorem which generalizes Theorems 1.7 and 1.10.

Theorem 1.11. *Let G be a $(k+2)$-connected graph where $k \geq 5$. If G contains three k-cliques, say L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 3k - 3$, then G contains a K_{k+2} as a minor.*

Note that the main theorem is for $k \geq 5$ since there are counterexamples to the theorem when $k = 3$ and $k = 4$ (while it is trivial that the theorem is true for $k = 1, 2$). Counterexamples for the case of $k = 3$ are 5-connected planar graphs. (Theorem 1.11 is true for non-planar graphs by Halin theorem ([6], or see p. 284 of [25]) in the case of $k = 3$.) Counterexamples for the case of $k = 4$ are apexes obtained from a 5-connected planar graph G' by adding a vertex adjacent to some vertices of G'. (Theorem 1.11 is true for non-apex graphs in the case of $k = 4$ by Theorem 1.7, Menger's theorem and the argument as in the 3.2.3.)

A k-connected graph may contain many $(k-2)$-cliques, but not necessary K_k-minor. For example, the graph $K_{k-5} + G_1$, where G_1 is a 5-connected planar graph, is K_k-minor-free and contains many copies of $(k-2)$-cliques. In this paper, Theorem 1.11, which generalizes Theorems 1.7 and 1.10, proves that a k-connected K_k-minor-free graph cannot contain three *"nearly" disjoint* $(k-2)$-cliques.

We hope our result would be used to prove some results on 7- and 8-chromatic graphs. In fact, in [9], Kawarabayashi proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor using our result. Maybe one can use this result to prove 8-chromatic case of Conjecture 1.9.

There is a conjecture by Seymour and Thomas (private communication with R. Thomas.)

Conjecture 1.12. *For every $p \geq 1$, there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)n - \frac{(p-1)(p-2)}{2} + 1$ edges has a K_p-minor.*

Note that the connectivity condition and the condition of the order of graphs are necessary because random graphs having no K_k-minor may have the average degree $k\sqrt{\log k}$, but all these graphs are small. So if a graph is large enough and highly connected, we do not know any construction of infinite family of counterexamples. This conjecture is true for $p \leq 9$. For $p \leq 7$, these were proved by Mader [12]. For $p = 8$, Jørgensen [7] proved. Very recently, Song and Thomas [18] proved the case $p = 9$. Note that all of these results do not require the connectivity condition in this conjecture.

We hope that our result could give a weaker result since, as far as we know, the only known extremal graphs are $K_{k-5} + G_1$, where G_1 is a 5-connected planar graph. So this graph could contain a $(k-2)$-clique. On the other hand, our result implies that it cannot contain three nearly "disjoint" K_{k-2}. Hence one can prove a weaker bound on the number of edges.

2. Terminology and notations

All graphs considered in this paper are finite, undirected, and without loops or multiple edges. The complete graph (or, clique, as a subgraph) on n vertices is denoted by K_n, and the complete bipartite graph such that one partite set has n vertices and the other partite set has m vertices is denoted by $K_{n,m}$. A circuit on n vertices is denoted by C_n. A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and vertices and contracting edges.

For a vertex x of a subgraph H_1 of G, the neighborhood of x in H_1 is denoted by $N_{H_1}(x)$. And, for a vertex $v \in V(G)$ and a vertex subset (or a subgraph) Y of G, $d_Y(x) = |\{v \in Y : xv \in E(G)\}|$. A graph G is *k-chromatic* if G is vertex-k-colorable but not vertex-$(k-1)$-colorable. Let V_1 and V_2 be subsets of $V(G)$. The symmetric difference of V_1 and V_2, denoted by $V_1 \Delta V_2$, is the set $(V_1 \cup V_2) - (V_1 \cap V_2)$.

3. Existence of a K_{k+2}-minor

The main theorem (Theorem 1.11) is to be proved in this section.

3.1. H-Wege lemma

The key lemma in our proof is Mader's "H-Wege" theorem which was proved in [13].

Lemma 3.1 ([13]). *Let G be a graph, let $S \subseteq V(G)$ be an independent set, and $k \geq 0$ be an integer. Then exactly one of the following two statements holds.*

(1) *There are k paths of G, each with two distinct ends both in S, such that each $v \in V(G) - S$ is in at most one of the paths.*
(2) *There exists a vertex set $W \subseteq V(G) - S$ and a partition $Y_1, \dots, Y_n$ of $V(G) - (S \cup W)$, and a subset $X_i \subseteq Y_i$, $1 \leq i \leq n$, such that*
   (a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor < k$,
   (b) *no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cap Y_i)$ and,*
   (c) *every path of $G - W$ with distinct ends both in S has an edge with both ends in Y_i for some i.*

Let $Z_1, Z_2, \dots, Z_h$ be subsets of $V(G)$. A path P of G with ends u, v is said to be *good* if there exist distinct i, j with $1 \leq i, j \leq h$ such that $u \in Z_i$ and $v \in Z_j$.

As Robertson et al. pointed out in [17], we can deduce the following lemma from Lemma 3.1.

Lemma 3.2 ([17]). *Let G be a graph, let $Z_1, Z_2, \dots, Z_h$ be subsets of $V(G)$, and let $k \geq 1$ be an integer. Then exactly one of the following two statements holds.*

(1) *There are k mutually disjoint good paths of G.*
(2) *There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, \dots, Y_n$ of $V(G) - W$, and a subset $X_i \subseteq Y_i$, for $1 \leq i \leq n$ such that*
   (a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor < k$,
   (b) *for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^{h} Z_j) \subseteq X_i$, and*
   (c) *every good path P in $G - W$ has an edge with both ends in Y_i for some i.*

3.2. Proof of the main theorem

Prove by way of contradiction. Assume G does not contain a K_{k+2} as a minor, and the following assertion is obvious by Menger's theorem.

3.2.1.

The graph G contains no clique of order $(k+1)$.

A path P of G with ends u, v is said to be *good* if there exist distinct i, j with $1 \leq i, j \leq 3$ such that $u \in L_i$ and $v \in L_j$. Let $L = L_1 \cup L_2 \cup L_3$.

3.2.2.

We claim that there do not exist $(k+2)$ mutually disjoint good paths in G.

Let $P_1, P_2, \ldots, P_{k+2}$ be a set of disjoint *good* paths of G. Let G' be the graph obtained by contracting P_i to a new vertex v_i for all $i \in \{1, 2, \ldots, k+2\}$. The subgraph Q of G' induced by v_i $(1 \leq i \leq k+2)$ is a K_{k+2}-clique and corresponds to a K_{k+2}-minor in G. □

3.2.3.

We claim that $|L_i \cap L_j - L_h| \leq 1$ **for every** $\{h, i, j\} = \{1, 2, 3\}$.

For otherwise, we may assume $|L_1 \cap L_2 - L_3| \geq 2$. Let $B \subseteq L_1 \cap L_2 - L_3$ with $|B| = 2$. Since $G - B$ is k-connected, there exist k disjoint good paths from L_3 to $L_1 \cup L_2 - B$, that implies that there exist $(k+2)$ mutually disjoint *good* paths in G. This contradicts 3.2.2. □

By Lemma 3.2 and 3.2.2, we have the following structure of G.

3.2.4.

There exists a vertex set $W \subseteq V(G)$ **and a partition** $Y_1, \ldots, Y_n$ **of** $V(G) - W$, **and a subset** $X_i \subseteq Y_i$, **for** $1 \leq i \leq n$ **such that**

(a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor \leq k+1$,

(b) **for any** i **with** $1 \leq i \leq n$, **no vertex in** $Y_i - X_i$ **has a neighbor in** $V(G) - (W \cup Y_i)$ **and** $Y_i \cap (\cup_{j=1}^{3} L_j) \subseteq X_i$, **and**

(c) **every good path** P **in** $G - W$ **has an edge with both ends in** Y_i **for some** i.

Let $M = (L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_3 \cap L_1)$, and choose W and $Y_1, X_1, \ldots, Y_n, X_n$ such that $|W|$ is as large as possible. Without loss of generality, we can assume that $Y_i \neq \emptyset$ for any $i \in \{1, 2, \ldots, n\}$. By the definition of W, M and 3.2.4(c), we have the following immediate observations.

3.2.5.

(a) $M \subseteq W$ by 3.2.4(c).

(b) $|L_1 \cup L_2 \cup L_3| = |L_1| + |L_2| + |L_3| - |M| - |L_1 \cap L_2 \cap L_3|$ by definition of M.

(c) $|M| + |L_1 \cap L_2 \cap L_3| \leq 3$ by the assumption $|L_1 \cup L_2 \cup L_3| \geq 3k - 3$.

(d) $|L_1 \cap L_2 \cap L_3| \leq 1$ by (c) and $L_1 \cap L_2 \cap L_3 \subseteq M$.

(e) $|L_i \cup L_j| \geq k+2$ **for** $1 \leq i < j \leq 3$. 3.2.5(e) is proved as follows: By 3.2.3 and 3.2.5(d), we have

$$
\begin{aligned}
|L_i \cup L_j| &= |L_i| + |L_j| - |L_i \cap L_j| \\
&= 2k - (|L_i \cap L_j \cap L_h| + |(L_i \cap L_j) - L_h|) \\
&\geq 2k - 2 = k + 2 + (k - 4) \geq k + 2
\end{aligned}
$$

where $\{i, j, h\} = \{1, 2, 3\}$. □

The following claim (f) follows from the assumption 3.2.4(b).

(f) $W \cup X_1 \cup \cdots \cup X_n \supseteq L_1 \cup L_2 \cup L_3$, **and** $|W| + \sum_{i=1}^{n} |X_i| \geq |L_1 \cup L_2 \cup L_3|$.

3.2.6.

By 3.2.5(c) and 3.2.5(d), there are only nine cases (illustrated in Figs. 1–9).

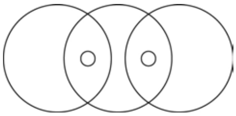

Fig. 1. (2, 0).

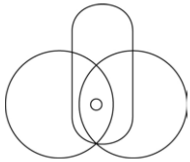

Fig. 2. (1, 1).

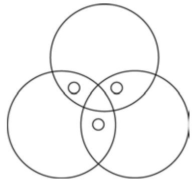

Fig. 3. (3, 0).

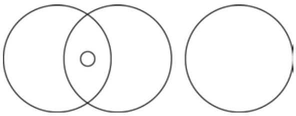

Fig. 4. (1, 0).

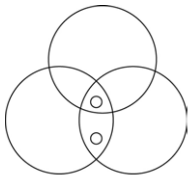

Fig. 5. (2, 1).

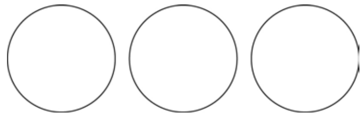

Fig. 6. (0, 0).

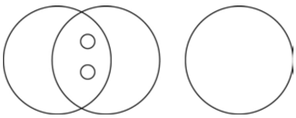

Fig. 7. (2, 0).

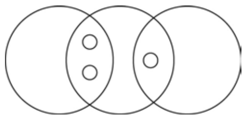

Fig. 8. (3, 0).

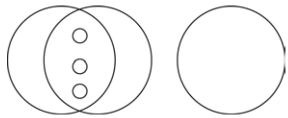

Fig. 9. (3, 0).

Legend for Figs. 1–9. $(i, j): \ i = |M|, \ j = |L_1 \cap L_2 \cap L_3|$.

Note that Figs. 7–9 are impossible by 3.2.3.

3.2.7.

We claim that $n \geq k-3$, **and if the equality holds then** $W = M$ **and** $|L_1 \cap L_2 \cap L_3| = 1$ **and** $L_1 \cup L_2 \cup L_3 = W \cup X_1 \cup \cdots \cup X_n$.

Since $|L_1 \cup L_2 \cup L_3| \geq 3k-3$ and $|W| \leq k+1$ (by 3.2.4(a)), we have $n \geq 1$. By 3.2.4(a), 3.2.5(a), (b), (d) and (f), we have

$$
\begin{aligned}
2(k+1) &\geq 2\left(|W| + \sum_{1 \leq i \leq n} \lfloor \tfrac{1}{2}|X_i| \rfloor\right) \geq 2|W| + \sum_{1 \leq i \leq n} |X_i| - n \\
&\geq |W| + |L_1 \cup L_2 \cup L_3| - n \geq |M| + |L_1 \cup L_2 \cup L_3| - n \\
&= |L_1| + |L_2| + |L_3| - |L_1 \cap L_2 \cap L_3| - n \geq 3k - 1 - n.
\end{aligned}
$$

Thus,

$$n \geq k-3$$

and if the equality holds then

$$|W| = |M| \qquad \text{and} \qquad |L_1 \cap L_2 \cap L_3| = 1$$

and $$|W| + \sum_{1\le i\le n} |X_i| = |L_1 \cup L_2 \cup L_3|.$$

3.2.8.

We claim that $X_i \neq \emptyset$ **for all** i.

Suppose that $X_i = \emptyset$ for some i. Then, since Y_i is not empty, W is a cutset and its cardinality is at most $k+1$ (by 3.2.4(a) and (b)). This contradicts that G is $(k+2)$-connected.

3.2.9.

We claim that $|X_i|$ **is odd for all** i.

Suppose that $|X_1|$ is even, then by 3.2.8, $|X_1| \geq 2$. Assume $v \in X_1$, let $W^* = W \cup \{v\}$, $Y_1^* = Y_1 - v$, $X_1^* = X_1 - v$ and $X_i^* = X_i, Y_i^* = Y_i$ for $2 \leq i \leq n$. Hence, the partition $\{W^*, X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*\}$ of $V(G)$ satisfies 3.2.4(a)–(c), contradicting the choice that $|W|$ is as large as possible. □

3.2.10.

Definition of A_i (**for** $i = 1, 2, 3$).

Let G'' be the subgraph obtained from $G - W$ by deleting all edges contained in any Y_j. Let A_i be the union of the vertex subsets of all components of G'' containing some vertex of L_i for each $i \in \{1, 2, 3\}$.

3.2.11.

Properties of $\{A_1, A_2, A_3\}$.

Properties of $\{A_1, A_2, A_3\}$ are to be studied in this subsection. The first property is immediate by 3.2.4 and the definition of A_i.

(a) $L_i - W \subseteq A_i \subseteq V(G) - W$ **for** $i = 1, 2, 3$.

Note that each $Y_j - X_j$ is an independent set of G'', and by 3.2.4(b), we have the following properties.

(b) $A_i \subseteq X_1 \cup \cdots \cup X_n$ **for** $i = 1, 2, 3$.

(c) A_1, A_2, A_3 **are disjoint** by the definition of A_i and 3.2.4(c).

(d) **Every path of** $G - W$ **from** A_i **to** A_{i^*} **has at least two vertices in** X_j **for some** j **and for** $1 \leq i, i^* \leq 3$ **with** $i \neq i^*$.

Proof of (d). Suppose there exists a path P from $v \in A_1$ to $u \in A_2$ in $G - W$. By the definition of A_1, A_2, we can take two disjoint paths Q and R such that Q is a path from some vertex $x \in L_1$ to v in $G - W$ and R is a path from some vertex $y \in L_2$ to u in $G - W$. Both Q and R have no edges with both ends in Y_j for any j. Then we have a path S from x to y by using P, Q, R. Since S is a good path by 3.2.4(c), S has an edge $e = x_1y_1 \in Y_j$ for some j. Note that $e \notin E(Q)$ and $e \notin E(R)$. This implies $e \in E(P)$ and $x_1, y_1 \in V(P)$. Note that, by 3.2.11(b), both v and u belong to $X_1 \cup \cdots \cup X_n$. By 3.2.4(b), the part of P from v to x_1 must contain a vertex from X_j, and likewise the part of P from y_1 to u. □

(e) $|A_i| \leq k + 1 - |W|$ **for** $1 \leq i \leq 3$.

Proof of (e). Suppose $|A_1| \geq k + 2 - |W|$. It is obvious that $|W| \leq k + 1$ (by 3.2.4(a)). Hence, $A_1 \neq \emptyset$. We also have that $L_2 \cup L_3 - W \neq \emptyset$ since $|L_2 \cup L_3| \geq k+2$ (by 3.2.5(e)) and $|W| \leq k + 1$ (by 3.2.4(a)).

Since $|L_2 \cup L_3| \geq k+2$ (by 3.2.5(e)), we have that $|L_2 \cup L_3 - W| \geq k+2-|W|$. Note that $G-W$ is $(k+2-|W|)$-connected; there are $(k+2-|W|)$ disjoint paths from A_1 to $L_2 \cup L_3 - W$ neither of which is empty. By 3.2.11(d), every path P_j contains at least two vertices of X_i for some i. Hence, $\sum_{1\leq i\leq n} \lfloor \frac{1}{2}|X_i| \rfloor \geq k+2-|W|$. This is a contradiction to 3.2.4(a). The other cases follow in a similar way. □

3.2.12.

We claim that $|W| \leq 3$.

This claim is to be proved in two steps in this subsection. First we show that

(a) $\sum_{i=1}^{3} |L_i \cap W| \leq |W| + 3$.

Note that $\sum_{i=1}^{3} |L_i \cap W| \leq |W|+|M|+|L_1 \cap L_2 \cap L_3|$. Hence, $\sum_{i=1}^{3} |L_i \cap W| \leq |W| + 3$ since $|M| + |L_1 \cap L_2 \cap L_3| \leq 3$ by 3.2.5(c). □

(b) By 3.2.11(a), (e) and 3.2.12(a), we have the following inequality:

$$\begin{aligned} 3k = \sum_{i=1}^{3} |L_i| &\leq \sum_{i=1}^{3}(|A_i| + |L_i \cap W|) \leq 3(k+1-|W|) + |W| + 3 \\ &= 3k + 6 - 2|W|. \end{aligned}$$

Hence, $|W| \leq 3$. □

3.2.13.

We claim that, for $1 \leq j \leq n$, **if** $|W \cup X_j| < (k+2)$ **then** $X_j = Y_j$.

Suppose that $X_j \neq Y_j$. Note that G is $(k+2)$-connected and by 3.2.4(b), $W \cup X_j$ is a vertex-cut separating $Y_j - X_j$ and $V(G) - Y_j - W$ neither of which is empty since $n \geq 2$ (by 3.2.7). It follows that $|W \cup X_j| \geq (k+2)$, as required. □

3.2.14.

We claim that, for $1 \leq j \leq n$, **if** $|X_j| \leq 3$ **then** $X_j = Y_j$.

By 3.2.13, it is obvious that $X_j = Y_j$ if $|X_j| \leq 3$ since $|W| \leq 3$ (by 3.2.12) and $k \geq 5$.

3.2.15.

Let $Z = (X_1 \cup \cdots \cup X_n) - (L_1 \cup L_2 \cup L_3)$.

3.2.16.

Some vertex-cuts of G.

Suppose that $X_i \cap L_j \neq \emptyset$ for some $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, 3\}$. By 3.2.4(c), 3.2.11(a) and (d), any path joining $X_i \cap L_j$ and $L_1 \cup L_2 \cup L_3 - W - L_j$ must use a vertex of W or Z or $X_i \Delta L_j$. Therefore, $(X_i \Delta L_j) \cup W \cup Z$ **is a cutset of** G **separating** $X_i \cap L_j$ **from** $L_1 \cup L_2 \cup L_3 - W - L_j$.

3.2.17.

We claim that $|X_i| \geq 3$ **for** $1 \leq i \leq n$.

This claim is to be proved in several steps in this subsection.

(a) First we show that, **for** $1 \leq i \leq 3$, $1 \leq j \leq n$, **if** $|X_j| = 1$, **then** $A_i \cap X_j = \emptyset$.

Suppose $A_1 \cap X_j \neq \emptyset$. Let $X_j = \{v\}$ and $N = N_G(v)$. Since G is $(k+2)$-connected, $|N| \geq k+2$. Hence $|N - W| \geq k+2-|W|$. Note that $|A_1| \leq k+1-|W|$

by 3.2.11(e); this implies $N - A_1 - W \neq \emptyset$. Take a vertex $x \in N - A_1 - W$. Since $|X_j| = 1$, we have $X_j = Y_j = \{v\}$ by 3.2.14. Note that $xv \in E(G)$, x is in A_1 by the definition of A_1, a contradiction. Hence $A_1 \cap X_j = \emptyset$. □

(b) Second we show that, **for** $1 \leq i \leq 3$, $1 \leq j \leq n$, **if** $|X_j| = 1$, **then** $A_i \cap N_G(X_j) = \emptyset$.

Suppose that $|X_1| = 1$ and $x \in A_1 \cap N_G(X_1)$. Hence, by 3.2.11(b), $x \in X_i$ for some $i \neq 1$. Since $|X_1| = 1$, by the definition of A_1 (defined in 3.2.10), $X_1 \subseteq A_1$. This contradicts 3.2.17(a) since $|X_1| = 1$. □

(c) Since $|X_i|$ is odd for each i (by 3.2.9), let m be an integer such that $m \leq n$ with $|X_i| = 1$ for $1 \leq i \leq m \leq n$ and $|X_j| \geq 3$ for $m + 1 \leq j \leq n$.

By the definition of A_i and 3.2.5, we have

$$\sum_{i=1}^{3} |A_i| \geq |L_1 \cup L_2 \cup L_3| - |W| = 3k - |M| - |L_1 \cap L_2 \cap L_3| - |W|. \quad \text{(I)}$$

Also, by 3.2.4(a),

$$\sum_{m+1 \leq j \leq n} |X_j| \leq 3 \sum_{m+1 \leq j \leq n} \lfloor \tfrac{1}{2}|X_j| \rfloor \leq 3 \sum_{1 \leq j \leq n} \lfloor \tfrac{1}{2}|X_j| \rfloor$$
$$\leq 3(k + 1 - |W|). \quad \text{(II)}$$

Assume $X = X_1 \cup X_2 \cup \cdots \cup X_m$ and $N = N_G(X)$. Then we can get the following.

(i) $N \subseteq W \cup X_{m+1} \cup \cdots \cup X_n$ by 3.2.4(b) and 3.2.14.
(ii) $N \cap A_1 = N \cap A_2 = N \cap A_3 = \emptyset$ by 3.2.17(b).
(iii) $|N| \geq k + 2$ since N separates X from $A_1 \cup A_2 \cup A_3$ (by 3.2.17(a) and (b)) and G is $(k + 2)$-connected.

Hence, we have

$$|N| + |A_1| + |A_2| + |A_3| \leq |W| + \sum_{i=m+1}^{n} |X_i|. \quad \text{(III)}$$

By (iii), (I)–(III) we have

$$(k + 2) + (3k - |M| - |L_1 \cap L_2 \cap L_3| - |W|) \leq |W| + 3(k + 1 - |W|)$$
$$= 3k + 3 - 2|W|.$$

Hence,

$$|W| \leq 1 + |M| + |L_1 \cap L_2 \cap L_3| - k.$$

By 3.2.5(a),

$$|W| \leq 1 + |W| + |L_1 \cap L_2 \cap L_3| - k.$$

That is, by 3.2.5(d),

$$k \leq 1 + |L_1 \cap L_2 \cap L_3| \leq 2.$$

This contradicts $k \geq 5$ and completes the proof of 3.2.17. □

3.2.18.

We prove some inequalities for $|Z|$.

(i)

$$|Z| \leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|,$$

and the equality holds if and only if $|X_j| = 3$ **for every** $j \in \{1, 2, \dots, n\}$.

(ii)

$$|Z| \leq 3 + |M| + |L_1 \cap L_2 \cap L_3| - 2|W|,$$

and the equality holds if and only if $|X_j| = 3$ **for every** $j \in \{1, 2, \dots, n\}$ **and** $W \subseteq L_1 \cup L_2 \cup L_3$.

Let $s = |Z|$. Then, by 3.2.5(f),

$$|X_1 \cup \cdots \cup X_n| = s + |L_1 \cup L_2 \cup L_3 - W|.$$

But, by 3.2.17, $|X_j| \leq 3\lfloor \frac{1}{2}|X_j| \rfloor$ for $1 \leq j \leq n$, and therefore

$$3 \sum_{1 \leq j \leq n} \lfloor \tfrac{1}{2}|X_j| \rfloor \geq \sum_{1 \leq j \leq n} |X_j| \geq s + |L_1 \cup L_2 \cup L_3 - W|,$$

with equality if and only if $|X_j| = 3$ for any $j \in \{1, 2, \dots, n\}$. By 3.2.4(a), we have

$$3(k + 1 - |W|) \geq s + |L_1 \cup L_2 \cup L_3 - W|.$$

That is,

$$s \leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|,$$

and the equality holds if and only if $|X_j| = 3$ for any $j \in \{1, 2, \dots, n\}$. This completes the proof of 3.2.18(i).

Note that, by 3.2.5(b), we have

$$\begin{aligned}|(L_1 \cup L_2 \cup L_3 - W)| &\geq |L_1 \cup L_2 \cup L_3| - |W| = 3k - |M| \\ &\quad - |L_1 \cap L_2 \cap L_3| - |W|,\end{aligned}$$

and the equality holds if and only if $W \subseteq L_1 \cup L_2 \cup L_3$. Hence, by 3.2.18(i),

$$\begin{aligned}s &\leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W| \leq 3k + 3 - 3|W| \\ &\quad - (3k - |M| - |L_1 \cap L_2 \cap L_3| - |W|) \\ &= 3 + |M| + |L_1 \cap L_2 \cap L_3| - 2|W|,\end{aligned}$$

and the equality holds if and only if $W \subseteq L_1 \cup L_2 \cup L_3$ and $|X_j| = 3$ for every $j \in \{1, 2, \dots, n\}$. This completes the proof of 3.2.18(ii). □

3.2.19.

(i) $|A_i \cap X_j| < \frac{1}{2}|X_j|$ **for** $1 \leq j \leq n$ **and** $1 \leq i \leq 3$.

Suppose that $|A_1 \cap X_1| \geq \frac{1}{2}|X_1|$. Since $X_1 \neq \emptyset$ by 3.2.8, there exists a vertex $v \in A_1 \cap X_1$. Since $|L_2 \cup L_3 - W| \geq |L_2 \cup L_3| - |W| \geq k + 2 - |W|$ by 3.2.5(e), and $G - W$ is $(k + 2 - |W|)$-connected, there are $(k + 2 - |W|)$ paths of $G - W$ between

A_1 and $L_2 \cup L_3 - W$, disjoint except possibly for v. Choose them with no internal vertex in A_1. By 3.2.11(d), each has at least two vertices in X_j for some $j \neq 1$, but at most $\lfloor \frac{1}{2}|X_j| \rfloor$ of them have two vertices in X_j for each $j \neq 1$. Note that by 3.2.4(a), we have

$$\sum_{2 \le j \le n} \lfloor \tfrac{1}{2}|X_j| \rfloor \le k + 1 - |W| - \lfloor \tfrac{1}{2}|X_1| \rfloor.$$

Thus, at least $1 + \lfloor \frac{1}{2}|X_1| \rfloor$ of them have two vertices in X_1. But each has only one vertex in A_1, and so has a vertex in X_1 which does not belong to A_1, and all these vertices in $X_1 - A_1$ are different. Hence $|X_1 - A_1| \ge 1 + \lfloor \frac{1}{2}|X_1| \rfloor$, a contradiction. □

(ii) $|L_i \cap X_j| < \frac{1}{2}|X_j|$ **for** $1 \le j \le n$ **and** $1 \le i \le 3$ **by**3.2.11(a) **and**3.2.19(i).

3.2.20.

(i) **We claim that if** $v \in A_i \cap X_j$ **for some** $i \in \{1, 2, 3\}$ **and some** $j \in \{1, 2, \dots, n\}$, **then** $d_{Y_j - A_i}(v) \ge 2$, **and the equality holds if and only if** $d_G(v) = k+2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ **and** $|A_i| = k + 1 - |W|$.

By the definition of A_i 3.2.10, we have

$$N_G(v) - (Y_j - A_i) \subseteq A_i \cup W - \{v\}.$$

Since G is $(k + 2)$-connected and $|A_i| \le k + 1 - |W|$ (by 3.2.11(e)), we have:

$$|N_G(v) \cap (Y_j - A_i)| \ge (k + 2) - |A_i \cup W - \{v\}| \ge (k + 2) - (k + 1 - |W| + |W| - 1) = 2$$

and the equality holds if and only if $d(v) = k + 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

(ii) We claim that **if** $v \in A_i \cap X_j$ **and** $|X_j| = 3$ **for some** $i \in \{1, 2, 3\}$ **and some** $j \in \{1, 2, \dots, n\}$, **then** $d_{X_j}(v) = 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ **and** $A_i = k + 1 - |W|$.

Note that $|X_j| = 3$. By 3.2.14, we have $Y_j = X_j$, and therefore,

$$d_{Y_j - A_i}(v) = d_{X_j - A_i}(v) \le 2.$$

On the other hand, by 3.2.20(i), we have $d_{Y_j - A_i}(v) \ge 2$. Hence $d_{Y_j - A_i}(v) = 2$. By 3.2.20(i) again, we are done.

3.2.21.

We claim that if $|X_j| = 3$ **for some** $j \in \{1, 2, \dots, n\}$ **then** $Z \cap X_j = \emptyset$.

For otherwise, we may assume $Z \cap X_i \neq \emptyset$, and let $x \in Z \cap X_i$. First we claim $x \in A_j$ for some $j \in \{1, 2, 3\}$. For otherwise, suppose $x \notin A_1 \cup A_2 \cup A_3$. Since $|X_i| = 3$, we have $X_j = Y_j$ by 3.2.14, and by the definition of A_i 3.2.10, we have $N_G(x) \subseteq W \cup Z \cup (X_j - \{x\})$. Note that, by 3.2.18(ii), 3.2.12, we have

$$|W| + |Z| + |X_j - \{x\}| \le |W| + (3 + |M| + |L_1 \cap L_2 \cap L_3| - 2|W|) + 2 = 5 + |M| + |L_1 \cap L_2 \cap L_3| - |W|.$$

Note that $|M| \le |W|$ and $|L_1 \cap L_2 \cap L_3| \le 1$ by 3.2.5(a) and 3.2.5(d). Hence, we have $|N_G(x)| \le 6$. This contradicts that G is $(k + 2)$-connected where $k \ge 5$.

Hence, without loss of generality, we may assume $x \in A_1$. By 3.2.20(ii), $W \cup A_1 \subseteq N_G(x) \cup \{x\}$. Note that $L_1 \subseteq A_1 \cup W$ by 3.2.11(a). Hence, $L_1 \subseteq N_G(x) \cup \{x\}$, since $x \in Z$, we have $x \notin L_1$. So $\{x\} \cup L_1$ induces a K_{k+1}-clique. This contradicts 3.2.1.

3.2.22.

We claim that if $|X_j| = 3$ **for some** j **then**

(1) $|X_j \cap A_i| = 1$ **for each** $i \in \{1, 2, 3\}$.
(2) X_j **induces a clique of** G.

By 3.2.21, $X_j \cap Z = \emptyset$, (1) follows by 3.2.19(i). (2) is an immediate corollary of 3.2.20(ii).

3.2.23.

We claim that there exists some $j \in \{1, 2, \ldots, n\}$ **such that** $|X_j| \geq 5$.

By 3.2.17, we may assume $|X_j| = 3$ for all $j \in \{1, 2, \ldots, n\}$. Hence, we have $X_j = Y_j$ by 3.2.14. There are two cases: $|Z| \neq 0$ and $|Z| = 0$.

Case 1. $|Z| \neq 0$. Since $Z \subseteq X_1 \cup X_2 \cup \cdots \cup X_n$ by the definition of Z, there exists X_j such that $X_j \cap Z \neq \emptyset$. This contradicts 3.2.21.

Case 2. $|Z| = 0$. By 3.2.22, we have $|A_i \cap X_j| = |L_i \cap X_j| = 1$, and X_j induces a clique of G. Let $v_{ij} \in L_i \cap X_j$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \ldots, n\}$. Furthermore, by 3.2.20(ii), $(W \cup A_1 \cup \{v_{2j}, v_{3j}\}) \subseteq N_G(v_{1j})$, hence, by contracting $L_2 - W$, $L_3 - W$ to a new vertex v, u respectively, then $L_1 \cup \{v, u\}$ induces a K_{k+2} minor. This is a contradiction.

3.2.24.

We claim that $|X_j| \geq 5$ **for any** $j \in \{1, 2, \ldots, n\}$.

For otherwise, by 3.2.17, we may assume $|X_1| = 3$. By 3.2.22(1), $|A_i \cap X_1| = 1$ for each $i \in \{1, 2, 3\}$. Hence, by 3.2.20(ii), $|A_i| = k + 1 - |W|$ for each $i \in \{1, 2, 3\}$.

Furthermore, by 3.2.11(b) and (c), we have

$$|Z| \geq |A_1| + |A_2| + |A_3| - |L_1 \cup L_2 \cup L_3 - W| = (3k + 3 - 3|W|) \\ - |L_1 \cup L_2 \cup L_3 - W|.$$

However, by 3.2.18(i), we have

$$|Z| = 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|.$$

The equality of 3.2.18(i) implies that $|X_i| = 3$ for all $i \in \{1, 2, \ldots, n\}$. This contradicts 3.2.23. □

3.2.25.

We show some inequalities for n.

By 3.2.24 and (3.2.4)(a),

$$5n \leq \sum_{1 \leq j \leq n} |X_j| \leq 2 * (k + 1 - |W|) + n = 2k + 2 + n - 2|W|. \tag{IV}$$

The inequality (IV) can be simplified as

$$2n \leq k + 1 - |W|. \tag{V}$$

Note that the equality of (IV) (and (V)), as well) implies that $|X_i| = 5$ for every i.

3.2.26.

We claim that $n = k - 3$.

For otherwise, since $n \geq k - 3$ by 3.2.7, we may assume that $n \geq k - 2$. By (V), we have

$$2k - 4 \leq 2n \leq k + 1 - |W|. \tag{VI}$$

That is,

$$k \leq 5 - |W|.$$

Note that $k \geq 5$. Hence, $|W| = 0$ and $k = 5$, and all equalities of (VI) hold, that is $n = k - 2 = 3$. By (IV), we have

$$15 \leq \sum_{1 \leq j \leq n} |X_j| \leq 2k + 2 + n - 2|W| = 15.$$

Therefore, the only possibility is $\{5, 5, 5\} = \{|X_1|, |X_2|, |X_3|\}$. Note that $|X_1| + |X_2| + |X_3| = |L_1| + |L_2| + |L_3|$ and $|W| = 0$ which implies $|L_i \cap L_j| = 0$ for $1 \leq i < j \leq 3$. Hence, $|Z| = 0$. By 3.2.19, $|L_i \cap X_1| < \frac{1}{2}|X_1|$ for $1 \leq i \leq 3$. Without loss of generality, we assume $|L_1 \cap X_1| = 2$. By 3.2.16, $(X_1 \Delta L_1) \cup W \cup Z = (X_1 \Delta L_1)$ is a cutset of G separating $X_1 \cap L_1$ from $L_2 \cup L_3$, and $|X_1 \Delta L_1| = 3 + 3 = 6 = k + 1$. It contradicts that G is $(k + 2)$-connected.

3.2.27.

The final step of the proof.

By 3.2.26, $n = k - 3$. By 3.2.7, we have

$$W = M \quad \text{and} \quad |L_1 \cap L_2 \cap L_3| = 1 \quad \text{and} \quad L_1 \cup L_2 \cup L_3 = W \cup X_1 \cup \cdots \cup X_n.$$

Hence,

$$|W| \geq 1 \qquad \text{and} \qquad Z = \emptyset. \tag{VII}$$

By (V) of 3.2.25, we have

$$2k - 6 = 2n \leq k + 1 - |W|.$$

That is,

$$k \leq 7 - |W|. \tag{VIII}$$

Note that $|W| \geq 3$ is impossible because $k \geq 5$. Therefore, there are only two cases: $|W| = 2$ and $|W| = 1$ (by (VII) and (VIII)).

Case 1. $|W| = 2$. In this case, $|W| = |M| = 2$, $|L_1 \cap L_2 \cap L_3| = 1$ (illustrated in Fig. 5), $k = 5$ and $n = k - 3 = 2$. Furthermore, the equality of (V) of 3.2.25 implies that

$$|X_1| = |X_2| = 5.$$

Without loss of generality, we assume $W \subseteq L_1$ and $|L_1 \cap X_1| = 2$. By 3.2.16, $(X_1 \Delta L_1)$ is a vertex-cut of order at most 6 since $Z = \emptyset$ and $W \subseteq L_1$. This contradicts that G is $(k+2)$-connected where $k = 5$.

Case 2. $|W| = 1$. In this case, $|W| = |M| = |L_1 \cap L_2 \cap L_3| = 1$ (illustrated in Fig. 2). Since

$$Z = \emptyset \qquad \text{and} \qquad |L_i \cap W| = |W| = 1$$

for each i, we have

$$\sum_{j=1}^{n} |X_j||L_1 \cup L_2 \cup L_3 - W| \sum_{j=1}^{n} |L_j - W|3k - 3. \tag{IX}$$

There are two subcases: $k = 6$ and $k = 5$ by (VIII).

Subcase 1. $k = 6$. In this subcase, $n = 3$ by 3.2.26. Hence, by (IX), we have

$$\sum_{j=1}^{3} |X_i| = 15.$$

Therefore, the only possibility in this subcase is $|X_1| = |X_2| = |X_3| = 5$ (by 3.2.24). Without loss of generality, we assume $|L_1 \cap X_1| = 2$. By 3.2.16, $(X_1 \Delta L_1)$ is a vertex-cut of order at most 7 since $Z = \emptyset$ and $W \subseteq L_1$. This contradicts that G is 8-connected.

Subcase 2. $k = 5$. In this subcase, $n = 2$ (by 3.2.26). By (IX),

$$\sum_{i=1}^{2} |X_i| = |L_1 \cup L_2 \cup L_3 - W| = 3k - 3 = 12.$$

Therefore, the only possibility in this subcase is that $|X_1| = 5$ and $|X_2| = 7$ (by 3.2.9 and 3.2.24).

Without loss of generality, we assume $|L_1 \cap X_1| = 2$. By 3.2.16, $(X_1 \Delta L_1)$ is a vertex-cut of order at most 6 since $Z = \emptyset$ and $W \subseteq L_1$. This contradicts that G is $(k+2)$-connected where $k = 5$.

This completes the proof of this theorem.

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