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MINIMAL PARTITIONS OF A BOX INTO BOXES

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A *box* is a set of the form $X=X_1\times\cdots\times X_d$, for some finite sets X_i, $i=1,\dots,d$. Answering a question posed by Kearnes and Kiss [2], Alon, Bohman, Holzman and Kleitman proved [1] that any partition of X into nonempty sets of the form $A_1\times\cdots\times A_d$, with $A_i\subsetneq X_i$, must contain at least 2^d members. In this paper we explore properties of such partitions with minimum possible number of parts. In particular, we derive two characterizations of *minimal partitions* among all partitions of X into *proper* boxes. For instance, let $P=P_1\times\cdots\times P_d$ be a fixed k*-dimensional plane* in X, that is $P_i=X_i$ for exactly k different subscripts i, with $|P_i|=1$ otherwise. It is shown that $\mathcal{F}$ is a minimal partition of X if and only if P intersects exactly 2^k members of $\mathcal{F}$, for every such P.

1. Introduction

Let $X=X_1\times\cdots\times X_d$ be the Cartesian product of finite sets X_i, $i=1,\dots,d$. If each X_i has more than one element, then X is called a d*-box*. Each nonempty subset A of X which itself is a Cartesian product, i.e. $A=A_1\times\cdots\times A_d$, with $A_i\subseteq X_i$, is called a *sub-box* of X (or simply a *box*). A box A is *proper* if $A_i\neq X_i$ for each $i=1,\dots,d$.

Problems considered in this paper emerged in connection with a recent paper by Kearnes and Kiss [2]. As a combinatorial counterpart of one of their results concerning an important class of *rectangular algebras*, they proved that in any partition of a d-box into at most d boxes none of the parts can be proper. And, occasionally, they asked if it is true that any partition of a d-box into less than 2^d boxes must contain a part which is not proper. This has

Mathematics Subject Classification (2000): 05A18, 52C22 been answered affirmatively by Alon, Bohman, Holzman and Kleitman [1]. An elegant argument provided in [1] uses an ingenious idea based on boxes with odd number of elements. There is also an apparently earlier electronic note [3] containing a slightly different solution (attributed to Bohman and Holzman) to the question of Kearnes and Kiss.

A superficial comparison of these two approaches shows that they are in a sense complementary. This observation is the starting point of the present investigation. However, our main aim is to discuss in detail the properties of *minimal partitions*, that is, partitions of a d-box X consisting of exactly 2^d proper boxes. In Section 3 we present two characterizations of minimal partitions among all partitions of X into proper boxes.

Another related problem concerns *spanning sets* of boxes. A set $\mathcal{S}$ of sub-boxes of X is called *spanning* if the minimal box containing all elements of $\mathcal{S}$ is X itself. Given a minimal partition of a d-box X, what is the minimal size of a spanning set consisting of members of this partition? This question is studied in Section 4. In Section 5 we describe a simple *recursive rule* for constructing all minimal partitions of a d-box X, starting with minimal partitions of a box of dimension $d-1$. The problem of finding such a rule has been raised in [1]. In Section 6 we show that if X is odd then there is a canonical one-to-one correspondence between members of a minimal partition of X and elements of the cube $\{0,1\}^d$. Finally, the last section of the paper is devoted to partitions of a box into arbitrary boxes.

2. Odd coverings

We start with a definition that appears in [3]. A box A is said to be *odd* if $|A|$ is an odd number. A covering $\mathcal{F}$ of a d-box X with proper boxes is said to be *odd* if for each $x \in X$ the number $|\{F \in \mathcal{F} : x \in F\}|$ is odd. In the sequel we always assume tacitly that every box in an odd covering of X is proper.

For a given integer $d \geq 1$ we write $[d] = \{1, \ldots, d\}$. By $I = \{i_1, \ldots, i_k\}$ we denote a (possibly empty) subset of $[d]$. If $I \subseteq [d]$ then its complement $[d] \setminus I$ is denoted by I'.

Given a box $A \subseteq X$ and $I \subseteq [d]$, let $A_I = A_{i_1} \times \cdots \times A_{i_k}$. In case of $I = \emptyset$ we adopt the convention that $A_\emptyset = \{\emptyset\}$. Also, for $i \in [d]$ we write simply $A_{i'}$ instead of $A_{\{i\}'}$. This notation is naturally extended to families of boxes.

A box $A \subseteq X$ is I-*odd* if A_I is odd and $A_{I'} = X_{I'}$. Finally, for a fixed $I \subseteq [d]$, we denote by $\mathcal{F}_I(A)$ the set of those $F \in \mathcal{F}$ for which $F_I \cap A_I$ is odd.

Our basic result reads as follows.

Theorem 1. *If $\mathcal{F}$ is an odd covering of a d-box X and B is an I-odd sub-box of X, with $|I|=k$, then*

$$|\mathcal{F}_I(B)| \geq 2^{d-k}.$$

Additionally, if $|\mathcal{F}|=2^d$, then

$$|\mathcal{F}_I(B)| = 2^{d-k}.$$

Proof. The method of the proof is a simple modification of that in [1] and [3]. Observe that the case $I=\emptyset$ corresponds to the original problem of Kearnes and Kiss. This will be the base for induction with respect to d in the first part of the proof, but we will verify it later.

Let $\mathbb{F}_2$ be the field of integers modulo 2. Clearly, each element of $\mathbb{F}_2^X$ may be identified with the characteristic function of a certain subset of X. Then the fact that $\mathcal{F}$ is an odd covering of X means simply that

$$1_X = \sum_{F\in\mathcal{F}} 1_F.$$

Let $I\neq\emptyset$ be fixed. To simplify notation assume that $I=\{1,\dots,k\}$. Consider now a mapping $T_B:\mathbb{F}_2^X\to\mathbb{F}_2^{X_{I'}}$ defined by the formula

$$T_B f(y) = \sum_{x:(x,y)\in B} f(x,y),$$

where $x\in X_I$, $y\in X_{I'}$ and (x,y) denotes the sequence x followed by y. Note that $T_X f=\sum_{x\in X} f(x)$, since it is natural to identify $\mathbb{F}_2^{\{\emptyset\}}$ with $\mathbb{F}_2$. Note also that T_B is an additive function.

Observe that since B is I-odd, for any box $A\subseteq X$ we have

$$T_B 1_A = \begin{cases} 1_{A_{I'}} & \text{if } A_I\cap B_I \text{ is odd,}\\ 0 & \text{otherwise.}\end{cases}$$

Hence, applying T_B to both sides of the expression for 1_X gives

$$1_{X_{I'}} = \sum_{F\in\mathcal{F}_I(B)} 1_{F_{I'}}.$$

The last equation means that the set $\{F_{I'}:F\in\mathcal{F}_I(B)\}$ is an odd covering of $X_{I'}$. By the induction hypothesis it has at least 2^{d-k} elements, which implies $|\mathcal{F}_I(B)|\geq 2^{d-k}$.

For the second assertion of the theorem, let $\mathcal{B}_I$ be the family of all I-odd boxes, and for $F\in\mathcal{F}$ let

$$\mathcal{B}_I(F) = \{B\in\mathcal{B}_I : F_I\cap B_I \text{ is odd}\}.$$

By the first part of the theorem

$$\sum_{B\in\mathcal{B}_I} |\mathcal{F}_I(B)| \geq |\mathcal{B}_I|\, 2^{d-k}.$$

Since $|\mathcal{B}_I(F)|=2^{-k}\,|\mathcal{B}_I|$, we obtain

$$2^{-k}\,|\mathcal{B}_I|\,|\mathcal{F}| = \sum_{F\in\mathcal{F}} |\mathcal{B}_I(F)| = \sum_{B\in\mathcal{B}_I} |\mathcal{F}_I(B)| \geq |\mathcal{B}_I|\, 2^{d-k}.$$

Clearly, if $|\mathcal{F}|=2^d$ the inequality above becomes an equality and the second assertion follows.

Let us note that the case $k=d$ holds independently of our induction argument. In fact, in this situation $|\mathcal{F}_I(B)|\geq 1$ means simply that there exists some $F\in\mathcal{F}$ such that $F\cap B$ is odd, which is obvious (compare [1]). Applying it to the last inequality gives $|\mathcal{F}|\geq 2^d$, which completes the induction. Actually, this is exactly the argument from [1]. ∎

Though a simple generalization of the result of [1] it is, Theorem 1 implies some interesting "geometrical" properties of minimal partitions. Consider a set $P\subseteq X$ such that P_I is a singleton and $P_{I'}=X_{I'}$. This is a special kind of an I-odd box which we call a *plane of dimension* $d-k$. In particular, P is a *hyperplane* if $k=1$, and a *line* if $k=d-1$. Clearly, if $\mathcal{F}$ is an odd covering of X, then the set $\mathcal{F}_I(P)$ consists of all boxes that intersect P. Thus we have the following corollary.

Corollary 1. *For each plane P of dimension $d-k$ in a d-box X and every odd covering $\mathcal{F}$ of X we have*

$$|\{F\in\mathcal{F} : F\cap P\neq\emptyset\}| \geq 2^{d-k}.$$

Moreover, if $\mathcal{F}$ is a minimal partition of X, then there are exactly 2^{d-k} members of $\mathcal{F}$ intersecting P.

It is justified to call the second part of this corollary the *equipartition principle*.

Another interesting consequence of Theorem 1 reads as follows.

Proposition 1. *If $\mathcal{F}$ is an odd covering of X of size 2^d, then for every pair $F,G\in\mathcal{F}$ there is $i\in[d]$ such that $\{F_i,G_i\}$ is a partition of X_i. In particular, $\mathcal{F}$ is a minimal partition of X.*

Proof. Suppose that for each i the set $\{F_i, G_i\}$ is not a partition of X_i. Two situations may occur: (1) $F_i \cap G_i \neq \emptyset$ or (2) $F_i \cap G_i = \emptyset$ but $F_i \cup G_i \neq X_i$. If (1) occurs we pick x_i from the common part of F_i and G_i and define $B_i = \{x_i\}$. Otherwise, we pick $u_i \in F_i$, $v_i \in G_i$ and $w_i \in X_i \setminus (F_i \cup G_i)$ and put $B_i = \{u_i, v_i, w_i\}$. A box $B = B_1 \times \cdots \times B_d$ obtained in this way is $[d]$-odd, hence, by Theorem 1, $\mathcal{F}_{[d]}(B)$ has only one element. But F and G are both in $\mathcal{F}_{[d]}(B)$, which is a contradiction. ∎

3. Minimal partitions

It is natural to expect that the properties of minimal partitions described above characterize them among all partitions of X with proper boxes. This intuition will be confirmed below.

Let $\mathcal{F}$ be a partition of a d-box X. We say that it satisfies the *splitting property* if for any two members $F, G \in \mathcal{F}$ there is $i \in [d]$ such that $\{F_i, G_i\}$ is a partition of X_i.

Theorem 2. *$\mathcal{F}$ is a minimal partition of a d-box X if and only if it satisfies the splitting property.*

Proof. By Proposition 1, only the "if" part remains to be proved. We use induction with respect to d. There is nothing to show if $d = 1$. If $d > 1$, fix $z \in X_d$ and put $H = X_{d'} \times \{z\}$. Let $\mathcal{H} = \{F \in \mathcal{F} : F \cap H \neq \emptyset\}$. Consider the related family $\mathcal{G} = \{F_{d'} : F \in \mathcal{H}\}$. Since $\{F_d, G_d\}$ can not be a partition of X_d, for every pair $F, G \in \mathcal{H}$, it follows that $\mathcal{G}$ inherits the splitting property of $\mathcal{F}$. Hence, $\mathcal{G}$ is a partition of $X_{d'}$, and, by the inductive assumption, it has 2^{d-1} elements. Consequently, $|\mathcal{H}| = 2^{d-1}$.

The same argument applies to $\mathcal{G}' = \{F_{d'} : F \in \mathcal{F} \setminus \mathcal{H}\}$. Thus, $\mathcal{F} \setminus \mathcal{H}$ also has 2^{d-1} elements, and the result follows. ∎

Yet another characterization of minimal partitions relates to the equipartition principle.

Theorem 3. *Let $k \in [d]$ be given. $\mathcal{F}$ is a minimal partition of a d-box X if and only if for every plane P of dimension k the family $\mathcal{P} = \{F \in \mathcal{F} : F \cap P \neq \emptyset\}$ contains exactly 2^k elements.*

Proof. As earlier, by Corollary 1 only the "if" part has to be shown. We reduce first the whole theorem to the case $k = 1$. To this end observe that each line L is contained in a certain plane P of dimension k. It is clear that P can be viewed as a k-box, and, by assumption, the family $\mathcal{E} = \{F \cap P : F \in \mathcal{P}\}$ is a partition of P consisting of 2^k proper boxes. Thus the equipartition principle applies to $\mathcal{E}$. In particular, L intersects exactly two elements of $\mathcal{E}$, so it does the same with $\mathcal{P}$.

For the case $k=1$ assume inductively that the assertion holds for boxes with dimension less than d. As in the preceding proof, consider families $\mathcal{H}=\{F\in\mathcal{F}: F\cap H\neq\emptyset\}$ and $\mathcal{G}=\{F_{d'}: F\in\mathcal{H}\}$, where $H=X_{d'}\times\{z\}$ is a fixed hyperplane. Clearly, $\mathcal{G}$ is a partition of $X_{d'}$ satisfying our inductive assumption. Hence,

$$|\mathcal{H}|=|\mathcal{G}|=2^{d-1}.$$

We will show now that also $\mathcal{G}'=\{F_{d'}: F\in\mathcal{F}\setminus\mathcal{H}\}$ is a partition of $X_{d'}$ satisfying the equipartition principle. First we will show that $\mathcal{G}'$ is a partition of $X_{d'}$. To this end suppose that there are two different boxes F and G in $\mathcal{F}\setminus\mathcal{H}$ such that $F_{d'}\cap G_{d'}\neq\emptyset$. Let x be their common element and let E be a box from $\mathcal{H}$ that contains the element (x,z). Then the line $L=\{x\}\times X_d$ intersects three boxes E,F,G, which is a contradiction. Thus, $\mathcal{G}'$ is a partition of $X_{d'}$. Moreover, $|\mathcal{F}\setminus\mathcal{H}|=|\mathcal{G}'|$.

To see that $\mathcal{G}'$ satisfies the inductive assumption suppose that there is a line K in $X_{d'}$ intersecting at least three boxes of $\mathcal{G}'$. Then a 2-dimensional plane $K\times X_d$ intersects at least five boxes of $\mathcal{F}$. There is a unique $i\in[d-1]$ such that $K_i=X_i$. Consider the family $\mathcal{D}=\{F_i\times F_d: F\in\mathcal{F}, F\cap K\times X_d\neq\emptyset\}$. Thus $|\mathcal{D}|\geq 5$. Arguing similarly as before we get that $\mathcal{D}$ is a partition of $X_i\times X_d$ such that any line of $X_i\times X_d$ intersects exactly two boxes of $\mathcal{D}$. However, in case of 2-dimensional boxes, this property is easily seen to contradict the fact that $|\mathcal{D}|\geq 5$. Hence, $\mathcal{G}'$ satisfies the inductive assumption and therefore $|\mathcal{G}'|=2^{d-1}$. This completes the proof. ∎

4. Spanning sets

In this section we consider another problem related to minimal partitions. Let $\mathcal{M}$ be a family of sub-boxes of a d-box X. We say that $\mathcal{M}$ is a *spanning set of boxes* if X is the minimal box containing all elements of $\mathcal{M}$. A natural question concerns the minimal size of a spanning set that may be found in a prescribed covering $\mathcal{F}$ of X. We denote this quantity by $S(\mathcal{F})$, and call it the *spanning number* of $\mathcal{F}$. In particular, we estimate it precisely for minimal partitions of X.

The following graph-theoretical lemma will be of use.

Lemma 1. *Let n and k be positive integers such that $n>2^{k-1}$. Suppose the edges of a complete graph K_n have been colored so that each color class forms a bipartite subgraph. Then there exists a multicolored (all edges have different color) tree $T\subseteq K_n$ with at least k edges.*

Proof. Assume $k>1$. We will construct a multicolored tree with k edges successively as follows. Choose any edge e of K_n and let c_e be its color. Consider the subgraph $G \subseteq K_n$ formed by all vertices of K_n and all edges in color c_e. Clearly, one of the bipartition classes of $V(G)$, say A, must have more than 2^{k-2} vertices. Then we pick an edge f with both ends in A, which is incident to e. Obviously, colors on edges spanned by A in K_n are different from c_e, hence $c_f \neq c_e$. The same argument may be repeated for f on the complete subgraph on A, thus obtaining the next edge with new color, and so on. This gives the desired tree of size k. ∎

Theorem 4. *Let $\mathcal{F}$ be a minimal partition of a d-box X and let $k \in [d]$. Then out of any $2^{k-1}+1$ members of $\mathcal{F}$ one can choose $k+1$ boxes $F^1,\dots,F^{k+1}$ such that $F^1_I,\dots,F^{k+1}_I$ span X_I, for some $I \subseteq [d]$ of size k. In particular, $S(\mathcal{F}) \le d+1$.*

Proof. Consider a complete graph Γ on the set of vertices $\mathcal{F}$ with edges colored in the following way. For an edge FG choose any color $i \in [d]$ such that the pair $\{F_i, G_i\}$ splits on X_i. By Proposition 1 all edges have been colored and certainly no odd cycle is monochromatic. Hence, any color class induces a bipartite graph and the assertion follows from Lemma 1. ∎

A special case of our next result shows that in general the above bound on $S(\mathcal{F})$ is optimal. In order to simplify its proof we find it useful to discuss first certain special partitions of 2-boxes.

Let Q be a nonempty set and let $\mathcal{Q}$ denote any of its partitions. The partition $\mathcal{Q}$ induces a new partition $\mathcal{Q}^*$ of $Q \times 2^{\mathcal{Q}}$ as follows. We take $K \times \mathcal{K}$ as a part of $\mathcal{Q}^*$ if and only if $K \in \mathcal{Q}$ and $\mathcal{K}$ is one of the two sets

$$\mathcal{K}^0 = \{\mathcal{A} \subseteq \mathcal{Q} : K \in \mathcal{A}\},\ \mathcal{K}^1 = \{\mathcal{A} \subseteq \mathcal{Q} : K \notin \mathcal{A}\}.$$

In the proof of the optimality of the bound on the spanning number given in Theorem 4 we will need the following lemma.

Lemma 2. *Let $\mathcal{P} \subseteq \mathcal{Q}^*$ and let $\mathbb{S} = \{\mathcal{K} \subseteq 2^{\mathcal{Q}} : K \times \mathcal{K} \in \mathcal{P}\}$. If $2^{\mathcal{Q}} = \bigcup \mathbb{S}$ then there is $D \in \mathcal{Q}$ such that both $D \times \mathcal{D}^0$ and $D \times \mathcal{D}^1$ belong to $\mathcal{P}$.*

Proof. The proof is a simple utilization of Cantor's diagonal method. Define

$$\mathcal{Z} = \{K \in \mathcal{Q} : K \times \mathcal{K}^0 \notin \mathcal{P} \text{ and } K \times \mathcal{K}^1 \in \mathcal{P}\}.$$

Since $\mathcal{Z} \in \bigcup \mathbb{S}$ there exists $B \times \mathcal{B} \in \mathcal{P}$ such that $\mathcal{Z} \in \mathcal{B}$. If we had $\mathcal{B} = \mathcal{B}^0$, then, by the definition of $\mathcal{Q}^*$, we would also have $B \in \mathcal{Z}$. In turn, this relation would imply, by the definition of $\mathcal{Z}$, that $B \times \mathcal{B}^0 \notin \mathcal{P}$, which is a contradiction. Thus $\mathcal{B} = \mathcal{B}^1$. So $B \notin \mathcal{Z}$, which is equivalent to saying that $B \times \mathcal{B}^0 \in \mathcal{P}$ or $B \times \mathcal{B}^1 \notin \mathcal{P}$. As the second possibility is excluded, we have $B \times \mathcal{B}^0 \in \mathcal{P}$. Henceforth, $D = B$ is an element the existence of which had to be shown. ∎

Proposition 2. *For each d there exists a d-box K^d with a minimal partition $\mathcal{F}^d$ satisfying the following property. Given arbitrary $I \subseteq [d]$ no $\mathcal{G} \subseteq \mathcal{F}^d$ can be found such that $|\mathcal{G}| = |I|$ and $\mathcal{G}_I$ spans K^d_I.*

Proof. We define inductively the sequence of pairs K^d, $\mathcal{F}^d$ for $d = 1, 2, \ldots$. For $d = 1$ put $K^1 = 2^{\{\emptyset\}}$ and $\mathcal{F}^d = \{\{\emptyset\}, \{\{\emptyset\}\}\}$. If $d \geq 2$ put $K^d = K^{d-1} \times 2^{\mathcal{F}^{d-1}}$ and $\mathcal{F}^d = (\mathcal{F}^{d-1})^*$ with $Q = K^{d-1}$ and $\mathcal{Q} = \mathcal{F}^{d-1}$.

Suppose that for some d and some $k \in [d]$ there exists $\mathcal{G}$ of cardinality k such that $\mathcal{G}_I$ spans K^d_I for some I of the same size. Take the smallest d and then the smallest k for which such a choice of $\mathcal{G}$ and I is possible. Obviously $d > 1$ and $k > 1$. Observe that d must belong to I, since otherwise $\mathcal{G}_I$ spans K^{d-1}_I, which contradicts the minimality of d. Now, by Lemma 2 there exists $D \in \mathcal{G}_{d'}$ such that $D \times \mathcal{D}^0$ and $D \times \mathcal{D}^1$ are disjoint members of $\mathcal{G}$. It follows that the cardinality of $\widetilde{\mathcal{G}} = \mathcal{G}_{d'}$ must be strictly smaller than that of $\mathcal{G}$. Moreover, $\widetilde{\mathcal{G}}_{\widetilde{I}}$ spans $K^{d-1}_{\widetilde{I}}$, where $\widetilde{I} = I \cap d'$. This contradicts the minimality of k, as $\left|\widetilde{\mathcal{G}}\right| \leq \left|\widetilde{I}\right| = k - 1$. ∎

Remark 1. It is natural to ask about the size of a minimal box K^d satisfying Proposition 2. Certainly, the above construction is not optimal with that respect. Another, more involved construction exists giving a bound of order $2^{O(d^3)}$.

5. Constructing minimal partitions

Let X be a d-box and let $Y = X_{d'}$. One may ask whether there are some rules according to which all minimal partitions $\mathcal{F}$ of X could be manufactured from those of Y. It is shown in [1] by means of an example that the simplest rule "*take a pair of minimal partitions $\mathcal{A}$ and $\mathcal{B}$ of Y and a partition $\{P, Q\}$ of X_d, and put $\mathcal{F} = (\mathcal{A} \times P) \cup (\mathcal{B} \times Q)$*" does not exhaust all possible patterns, even if d is allowed to be replaced by any $i \in [d]$. However, modifying slightly the above approach we obtain the appropriate procedure.

Let $Y = X_{d'}$ and let $\mathcal{K}$ be the finest partition of Y such that each $K \in \mathcal{K}$ is a union of some members of $\mathcal{A}$ as well as some members of $\mathcal{B}$. Obviously, K need not be a box. For a given K fix any partition $\{A(K), B(K)\}$ of X_d and define a minimal partition $\mathcal{D}$ of X as follows. A box D is a part of $\mathcal{D}$ if and only if there is $A \in \mathcal{A}$ such that $D = A \times A(K)$ and $A \subseteq K$ or there is $B \in \mathcal{B}$ such that $D = B \times B(K)$ with $B \subseteq K$.

Conversely, if we start with a minimal partition $\mathcal{D}$ of X then it can be easily split into two minimal partitions $\mathcal{A}$ and $\mathcal{B}$ of Y. It suffices to choose a hyperplane $H = Y \times \{z\}$ and define $\mathcal{A}$ and $\mathcal{B}$ as families of projections of boxes intersecting H and disjoint with H, respectively.

6. Even-odd pattern

Let us suppose that A is a sub-box of a d-box X and let $I \subseteq [d]$. We say that I is the *even-odd pattern* of A if the set A_i has an even number of elements for $i \in I$ and an odd number of elements for $i \in I'$. If A is a proper sub-box of X then the *I-complement* of A, denoted by A^I, is defined by the formula

$$A^I = \{x \in X : x_i \notin A_i \text{ for } i \in I, \text{ and } x_i \in A_i \text{ otherwise}\}.$$

If $\mathcal{F}$ is a minimal partition of X then $\mathcal{F}^I = \{A^I : A \in \mathcal{F}\}$ is called the *I-complement* of $\mathcal{F}$. As before we simplify our notation in case of $I = \{i\}$ by dropping the brackets.

Lemma 3. *If $\mathcal{F}$ is a minimal partition of a d-box X then $\mathcal{F}^I$ is also a minimal partition of X.*

Proof. Let $I = \{i_1, \ldots, i_k\}$. It is clear that $\mathcal{F}^I = (\ldots(\mathcal{F}^{i_1})^{i_2}\ldots)^{i_k}$. Hence, it suffices to show that for any $i \in [d]$ the i-complement of $\mathcal{F}$ is a minimal partition of X.

Suppose for simplicity that $i = d$ and let F and G be two different parts of $\mathcal{F}$. The splitting property of $\mathcal{F}$ implies easily that their complements F^d and G^d are disjoint. Thus it remains to be shown that $\mathcal{F}^d$ covers X. Let $x \in X$ and let $A \in \mathcal{F}$ be the unique part such that $x \in A$. For any z chosen from the set $X_d \setminus A_d$ there exists $B \in \mathcal{F}$, such that $(x_{d'}, z) \in B$. Therefore $A \neq B$ and these two sets are disjoint. In particular, $x \notin B$ which means that $x \in B^d$. ∎

Using this Lemma we derive the following uniqueness property of even-odd patterns in minimal partitions of odd boxes.

Theorem 5. *If a d-box X is odd and $\mathcal{F}$ is its minimal partition then for any $I \subseteq [d]$ there exists exactly one $A \in \mathcal{F}$ for which I is its even-odd pattern.*

Proof. If we let $B = X$ in Theorem 1 then we deduce with the aid of Lemma 3 that there is a unique box $C \in \mathcal{F}^I$ which is odd. Consequently, $A = C^I$ is the unique element of $\mathcal{F}$ for which I is an even-odd pattern. ∎

7. Partitions into arbitrary boxes

We consider now partitions of a d-box X into boxes that are not necessarily proper. A box A is said to be k-proper if there exists $I \subseteq [d]$ of size k such that $A_{I'} = X_{I'}$ and A_I is proper in X_I.

We say that a partition $\mathcal{F}$ of X is *nontrivial* if it consists of at least two boxes. Assuming $\mathcal{F}$ to be nontrivial denote by p_k the number of all k-proper boxes in $\mathcal{F}$. We say that the *multi-index* $p = (p_1, \ldots, p_d)$ is associated with $\mathcal{F}$.

It is easy to observe that q is a multi-index with a certain nontrivial partition of $\{0,1\}^d$ into boxes if and only if

$$q_1 2^{d-1} + q_2 2^{d-2} + \cdots + q_d = 2^d.$$

Proposition 3. *Let p be the multi-index of a nontrivial partition $\mathcal{F}$ of X. Then there is a multi-index q associated with certain nontrivial partition of $\{0,1\}^d$ such that $p \geq q$ in the coordinate order. In particular,*

$$p_1 2^{d-1} + p_2 2^{d-2} + \cdots + p_d \geq 2^d.$$

Proof. Each k-proper box $A \in \mathcal{F}$ can be split into 2^{d-k} proper boxes. Thus we can generate from $\mathcal{F}$ a new partition $\mathcal{G}$ consisting of only proper boxes such that $|\mathcal{G}| = p_1 2^{d-1} + p_2 2^{d-2} + \cdots + p_d$. On the other hand, we know that $|\mathcal{G}| \geq 2^d$. So, by an easy arithmetic argument the existence of the desired q follows. ∎

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