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CIRCULAR CHROMATIC NUMBER AND MYCIELSKI GRAPHS

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As a natural generalization of graph coloring, Vince introduced the star chromatic number of a graph G and denoted it by $\chi^*(G)$. Later, Zhu called it circular chromatic number and denoted it by $\chi_c(G)$. Let $\chi(G)$ be the chromatic number of G. In this paper, it is shown that if the complement of G is non-hamiltonian, then $\chi_c(G)=\chi(G)$. Denote by $M(G)$ the Mycielski graph of G. Recursively define $M^m(G)=M(M^{m-1}(G))$. It was conjectured that if $m\le n-2$, then $\chi_c(M^m(K_n))=\chi(M^m(K_n))$. Suppose that G is a graph on n vertices. We prove that if $\chi(G)\ge \frac{n+3}{2}$, then $\chi_c(M(G))=\chi(M(G))$. Let S be the set of vertices of degree $n-1$ in G. It is proved that if $|S|\ge 3$, then $\chi_c(M(G))=\chi(M(G))$, and if $|S|\ge 5$, then $\chi_c(M^2(G))=\chi(M^2(G))$, which implies the known results of Chang, Huang, and Zhu that if $n\ge 3$, $\chi_c(M(K_n))=\chi(M(K_n))$, and if $n\ge 5$, then $\chi_c(M^2(K_n))=\chi(M^2(K_n))$.

1. Introduction

All graphs considered are finite and simple. Let k and d be positive integers such that $k\ge 2d$. A (k,d)-*coloring* of a graph G with vertex-set $V(G)$ and edge-set $E(G)$ is a mapping $c\colon V(G)\to\{0,1,\dots,k-1\}$ such that for each edge $xy\in E(G)$, $d\le |c(x)-c(y)|\le k-d$. A $(k,1)$-coloring of G is simply a proper k-coloring of G. As a natural generalization of the chromatic number,Vince [6] introduced the *star chromatic number* of a graph G, which is defined to be the infimum of the ratio k/d for which G has a (k,d)-coloring. It was shown [6] that the infimum can be replaced by minimum. Zhu [7] equivalently

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considered a (k,d)-coloring of G as a mapping c from $V(G)$ to open arcs of unit length in a circle of length k/d such that for each edge $xy \in E(G)$, $c(x) \cap c(y) = \emptyset$. Instead of "star chromatic number", Zhu used "circular chromatic number" and denoted it by $\chi_c(G)$ (see [8]). In this paper, we use Zhu's notations, but for the definition of circular chromatic number, we still use the one given by Vince [6] (or see Bondy and Hell [2]).

Definition 1.1. Let G be a graph on n vertices. The *circular chromatic number* of G is defined as $\chi_c(G) = \inf\{\frac{k}{d} :\ G \text{ has a } (k,d)\text{-coloring}\}$.

Let G_k^d denote the graph whose vertex-set is $\{0,1,2,\ldots,k-1\}$ and whose edge-set is $\{ij : d \le |i-j| \le k-d\}$. Vince [6] showed that $\chi_c(G_k^d) = \frac{k}{d}$. (An easier proof was given by Bondy and Hell [2].) For two graphs G and H, a *homomorphism* from G to H is a mapping $f:\ V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. It is easy to show (see [2]) that G has a (k,d)-coloring if and only if there is a homomorphism from G to G_k^d. The following proposition was obtained by Zhu [7]. Here we provide an easy derivation from a result of Bondy and Hell [2].

Proposition 1.2. *Let f be a homomorphism from G to G_k^d, where* $\gcd(k,d)=1$. *If f is not surjective, then $\chi_c(G) < \frac{k}{d}$.*

Proof. Since f is not surjective, it is a homomorphism from G to a subgraph $H = G_k^d - v$ for some vertex $v \in V(G_k^d)$. Note that $V(H) \subseteq V(G_k^d) = \{0,1,\ldots,k-1\}$. Define $c(i) = i$ for each vertex $i \in V(H)$. Then c is a (k,d)-coloring of H. Since $|V(H)| < |V(G_k^d)| = k$, by [2, Proposition 2], H has a (t,s)-coloring with $\frac{t}{s} < \frac{k}{d}$. Let g be a homomorphism from H to G_t^s. The composition of f and g is a homomorphism from G to G_t^s, which implies $\chi_c(G) \le \frac{t}{s} < \frac{k}{d}$, as required. ∎

In this paper we consider a (k,d)-coloring of a graph G as a partition of $V(G)$. A *(k,d)-partition* of G is a partition $(X_0, X_1, \ldots, X_{k-1})$ of $V(G)$ such that for each j, $0 \le j \le k-1$,

$$X_j \cup X_{j+1} \cup \ldots \cup X_{j+d-1}$$

is an independent set in G, where the addition of indices is taken mod k. (Here it is allowed that $X_i = \emptyset$.) It is easy to see that a (k,d)-partition of G is simply the color classes of a (k,d)-coloring of G. Thus we have the following easy observation.

Observation 1.3. *A graph G has a (k,d)-coloring if and only if it has a (k,d)-partition.*

Proposition 1.4. *Let $(X_0, X_1, \ldots, X_{k-1})$ be a (k,d)-partition of G, where $\gcd(k,d)=1$. If $X_t=\emptyset$ for some t, then $\chi_c(G)<\frac{k}{d}$.*

Proof. Let $V(G_k^d)=\{0,1,\ldots,k-1\}$. Define a homomorphism f from G to G_k^d by $f(v)=i$ if $v\in X_i$. Since $X_t=\emptyset$, we have that $f^{-1}(t)=\emptyset$, and hence f is not surjective. By Proposition 1.2, $\chi_c(G)<\frac{k}{d}$, as required. ∎

2. Sufficient Conditions for Graphs G with $\chi_c(G)=\chi(G)$

As usual, let $\chi(G)$ denote the chromatic number of a graph G. It is shown by Vince [6] that

$$\chi(G)-1<\chi_c(G)\le\chi(G).$$

What determines whether $\chi_c(G)=\chi(G)$? This problem was raised by Vince [6] and has been investigated extensively. Abbott and Zhou [1] proved that if the complement of G is disconnected, then $\chi_c(G)=\chi(G)$. A special case of this result is that if G has a vertex of degree $|V(G)|-1$, then $\chi_c(G)=\chi(G)$, a result which was previously proved by Zhu [7] and by Guichard [4]. In this section we show that if the complement of G is non-hamiltonian, then $\chi_c(G)=\chi(G)$. Since a necessary condition for a graph to be hamiltonian is to be 2-connected, an immediate consequence of our result is that if the complement of G is not 2-connected, then $\chi_c(G)=\chi(G)$. This improves the earlier results mentioned above.

Theorem 2.1. *If $\chi_c(G)<\chi(G)$, then the complement of G is hamiltonian.*

Proof. Let $\chi_c(G)=\frac{k}{d}$, where $d\ge 2$ since $\chi_c(G)<\chi(G)$. By [2, Corollary 1], we may assume that $\gcd(k,d)=1$. By Observation 1.3, G has a (k,d)-partition $\{X_0, X_1, \ldots, X_{k-1}\}$, and by Proposition 1.4, $X_i\neq\emptyset$, $0\le i\le k-1$. Since $d\ge 2$, $X_i\cup X_{i+1}$ is an independent set in G, $0\le i\le k-1$, where $X_k=X_0$. Thus, in the complement of G, X_i induces a complete subgraph and each vertex of X_i is adjacent to every vertex of X_{i+1}, $0\le i\le k-1$. It follows that the complement of G has a hamiltonian cycle in which the vertices of each X_i are consecutive on the hamiltonian cycle. ∎

Let C be a cycle. The *dth power* of C is the graph obtained from C by adding edges joining every pair of vertices with distance at most d in C. (The 1st power of C is simply C itself.) It is easy to see that the arguments used in the proof of Theorem 2.1 give the following more general result.

Theorem 2.2. *If $\chi_c(G)=\frac{k}{d+1}$, where $d\ge 1$ and $\gcd(k,d+1)=1$, then the complement of G contains the dth power of a hamiltonian cycle.*

A proper k-coloring of G is simply a $(k,1)$-partition of G, that is, a partition of $V(G)$ into $(V_1,V_2,\dots,V_k)$ such that each V_i is an independent set, $1\le i\le k$. Another application of (k,d)-partitions is the following easier proof of a result of Steffen and Zhu [5].

Theorem 2.3. *Let $\chi(G)=t$. If there is a nonempty proper subset A of $V(G)$ such that for any t-coloring c of G, and for any color class F of c, either $F\subseteq A$ or $F\cap A=\emptyset$, then $\chi_c(G)=t$.*

Proof. If not true, let $\chi_c(G)=\frac{k}{d}$ with $d\ge 2$ and $\gcd(k,d)=1$. As before, G has a (k,d)-partition $\{X_0,X_1,\dots,X_{k-1}\}$ and for each j, $0\le j\le k-1$,

$$F_j = X_j\cup X_{j+1}\cup\ldots\cup X_{j+d-1}$$

is an independent set. Let $m=\lfloor\frac{k}{d}\rfloor$ and set

$$F_{md} = X_{md}\cup X_{md+1}\cup\ldots\cup X_{k-1}.$$

Then $(F_0,F_d,\dots,F_{md})$ is an $(m+1)$-coloring of G. Note that $m+1=\lceil\frac{k}{d}\rceil=t$. By the given condition, for each i, $0\le i\le m$, either $F_{id}\subseteq A$ or $F_{id}\cap A=\emptyset$, which implies that there is an s such that $F_{sd}\subseteq A$ and $F_{(s+1)d}\subseteq V(G)\setminus A$, where $F_{(m+1)d}=F_0$. Then $(F_1,F_{d+1},\dots,F_{md+1})$ is a t-coloring of G in which neither $F_{sd+1}\subseteq A$ nor $F_{sd+1}\cap A=\emptyset$, where $F_{md+1}=(F_{md}\setminus X_{md})\cup X_0$. This contradiction completes the proof. ∎

For the next two results, and for the use in the next section, we need some additional notations. Let v be a vertex in a graph G. A *neighbor* of v is a vertex that is adjacent to v. Denote by $N(v)$ the set of neighbors of v. Then $d(v)=|N(v)|$ is the *degree* of v. If H is a subgraph of G, define

$$N(H)=\bigcup_{v\in V(H)} N(v) \quad\text{and}\quad d(H)=\sum_{v\in V(H)} d(v).$$

Proposition 2.4. *If $\chi_c(G)=\frac{k}{d}$, where $\gcd(k,d)=1$, then $d(v)\le |V(G)|-2d+1$ for each $v\in V(G)$.*

Proof. Let $\{X_0,X_1,\dots,X_{k-1}\}$ be a (k,d)-partition of G. For any $v\in V(G)$, say $v\in X_j$, let

$$A=X_j\cup X_{j+1}\cup\ldots\cup X_{j+d-1} \text{ and } B=X_j\cup X_{j-1}\cup\ldots\cup X_{j-d+1}.$$

So A and B are independent sets. Note that $A\cap B=X_j$, we see that v is not adjacent to any vertex of X_i for each $i\in\{j+d-1,\dots,j+1,j,j-1,\dots,j-d+1\}$. Since $X_i\ne\emptyset$ for each i by Proposition 1.4, it follows that v is not adjacent to at least $2d-1$ vertices in G, that is, $d(v)\le|V(G)|-2d+1$, as required. ∎

Proposition 2.4 is best possible in the sense that $\chi_c(G_k^d) = \frac{k}{d}$, while $d(v) = |V(G_k^d)| - 2d + 1$ for each $v \in V(G_k^d)$. (Proposition 2.4 can also be derived from Theorem 2.2.) We conclude this section with the following result which has applications to Mycielski graphs in the next section.

Theorem 2.5. *Let G be a graph on n vertices. If there is a set S of three vertices in G such that $N(S) \cup S \neq V(G)$ and $d(S) \geq 3(n-3)$, then $\chi_c(G) = \chi(G)$.*

Proof. Suppose, to the contrary, that $\chi_c(G) = \frac{k}{d} < \chi(G)$, where $d \geq 2$ and $\gcd(k,d) = 1$. Let $\{X_0, X_1, \ldots, X_{k-1}\}$ be a (k,d)-partition of G. Let $x \in S$, say $x \in X_j$ for some j. Since $d \geq 2$, we have that $X_j \cup X_{j-1}$ and $X_j \cup X_{j+1}$ are independent sets, and hence $N(x) \subseteq V(G) \setminus (X_{j-1} \cup X_j \cup X_{j+1})$, where the addition of indices is taken mod k. By Proposition 1.4, $X_i \neq \emptyset$, $0 \leq i \leq k-1$, and therefore $d(x) \leq n-3$ with equality only if $N(x) = V(G) \setminus (X_{j-1} \cup X_j \cup X_{j+1})$ and $|X_{j-1}| = |X_j| = |X_{j+1}| = 1$. Let $S = \{x_p, x_q, x_r\}$. Then

$$3(n-3) \leq d(S) = d(x_p) + d(x_q) + d(x_r) \leq 3(n-3),$$

which implies that for each $i \in \{p,q,r\}$, equality $d(x_i) = n-3$ holds, and thus $N(x_i) = V(G) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ and $|X_{i-1}| = |X_i| = |X_{i+1}| = 1$. Noting that $N(S) = N(x_p) \cup N(x_q) \cup N(x_r)$, we have that $N(S) \cup S = V(G)$, which is contrary to the given condition and completes the proof. ∎

3. Mycielski Graphs

Let G be a graph with $V(G) = \{x_i : 1 \leq i \leq n\}$. The *Mycielski graph* of G, denoted by $M(G)$, is the graph obtained from G by adding $n+1$ new vertices $x'_1, x'_2, \ldots, x'_n, u$, and then, for $1 \leq i \leq n$, joining x'_i to the neighbors of x_i and to u. The vertex x'_i is called the *twin* of x_i (x_i is also the twin of x'_i). The vertex u is called the *root* of $M(G)$. It is well known that for any nonempty graph G, $\chi(M(G)) = \chi(G) + 1$. However, there are infinitely many graphs G for which $\chi_c(M(G)) \neq \chi_c(G) + 1$. For instance, let G be a cycle of length $2m+1$. It is known [6] that $\chi_c(G) = 2 + \frac{1}{m}$. But, by a result in [3], $\chi_c(M(G)) = 4$. Let K_n denote the complete graph on n vertices. Chang, Huang, and Zhu [3] proved that if $n \geq 3$, then $\chi_c(M(K_n)) = \chi_c(K_n) + 1 = \chi(M(K_n))$. Their proof is rather involved. We present here a stronger result with an easier proof.

Theorem 3.1. *Let G be a graph on n vertices and K the set of vertices of degree $n-1$. If $|K| \geq 3$, then $\chi_c(M(G)) = \chi_c(G) + 1 = \chi(M(G))$.*

Proof. Let $S \subseteq K$ with $|S| = 3$. For each $v \in S$, since v has degree $n-1$ in G, and by the structure of $M(G)$, v has degree $2(n-1) = |V(M(G)| - 3$ in $M(G)$. Therefore, in $M(G)$, $d(S) = 3(|V(M(G)| - 3)$. Clearly, the root of $M(G)$ is not in $N(S) \cup S$. Applying Theorem 2.5 to $M(G)$, we have that $\chi_c(M(G)) = \chi(M(G)) = \chi(G) + 1$. Since G has vertices of degree $n-1$, and by Proposition 2.4, $\chi(G) = \chi_c(G)$. This completes the proof. ∎

Theorem 3.2. *Let G be a graph on n vertices. If $\chi_c(M(G)) = \frac{k}{d}$, where $\gcd(k,d) = 1$, then $\chi(G) < 2 + \frac{n-1}{d}$.*

Proof. Let $V(M(G)) = \{x_1, x_2, \ldots, x_n; x'_1, x'_2, \ldots, x'_n; u\}$, where $V = \{x_1, x_2, \ldots, x_n\}$ is the vertex-set of G, $V' = \{x'_1, x'_2, \ldots, x'_n\}$ is the set of the twins of vertices of V, and u is the root of $M(G)$. Let $(X_0, X_1, \ldots, X_{k-1})$ be a (k,d)-partition of $M(G)$. We may assume that $u \in X_0$, and set

$$A = X_{d-1} \cup X_{d-2} \cup \ldots \cup X_1 \cup X_0 \cup X_{k-1} \cup \ldots \cup X_{k-d+2} \cup X_{k-d+1}.$$

We note that $A \subseteq V$ since $u \in X_0$. Let $x \in V \setminus A$, and denote its twin by x', say $x \in X_j$ $(d \le j \le k-d)$. If $x' \notin X_j$, say $x' \in X_\ell$ with $\ell \ne j$, we may move x' to X_j to obtain a new (k,d)-partition $(Y_0, Y_1, \ldots, Y_{k-1})$, where $Y_i = X_i$ if $i \notin \{j, \ell\}$, $Y_j = X_j \cup \{x'\}$, and $Y_\ell = X_\ell \setminus \{x'\}$. Now, x and x' lie in the same Y_j in the new (k,d)-partition. By this argument, we may assume that the (k,d)-partition $(X_0, X_1, \ldots, X_{k-1})$ has been chosen such that for any $x \in V \setminus A$,

$$\{x, x'\} \subseteq X_j \text{ for some } j, \ d \le j \le k-d, \tag{3.1}$$

and by Proposition 1.4, $X_i \ne \emptyset$, $0 \le i \le k-1$. Let $A' = \{x'_i : \ x_i \in A\}$. Then for each j, $d \le j \le k-d$, X_j contains either a vertex in $V \setminus A$ or a vertex in A'. Using $|A'| = |A|$, it follows that

$$k - 2d + 1 \le |V \setminus A| + |A'| = |V| = n.$$

So $k \le n + 2d - 1$, and hence $\chi_c(M(G)) = \frac{k}{d} \le 2 + \frac{n-1}{d}$. But, $\chi_c(M(G)) > \chi(M(G)) - 1 = \chi(G)$, and therefore, $\chi(G) < 2 + \frac{n-1}{d}$, as required. ∎

Corollary 3.3. *Let G be a graph on n vertices. If $\chi(G) \ge \frac{n+3}{2}$, then $\chi_c(M(G)) = \chi(M(G))$.*

Proof. If $\chi_c(M(G)) < \chi(M(G))$, let $\chi_c(M(G)) = \frac{k}{d}$, where $d \ge 2$ and $\gcd(k,d) = 1$. By Theorem 3.2, $\chi(G) < 2 + \frac{n-1}{d} \le \frac{n+3}{2}$, a contradiction. ∎

If $n \ge 3$, then $\chi(K_n) = n \ge \frac{n+3}{2}$ so that Theorem 3.2 and Corollary 3.3 provide another route to the result of Chang, Huang, and Zhu [3] that if $n \ge 3$, then $\chi_c(M(K_n)) = \chi(M(K_n))$. Let $\omega(G)$ denote the *clique number* of

G (the number of vertices in a maximum complete subgraph). It is known that $\chi(G) \geq \omega(G)$. Thus, another easy consequence of Corollary 3.3 is that if $\omega(G) \geq \frac{|V(G)|+3}{2}$, then $\chi_c(M(G)) = \chi(M(G))$.

Theorem 3.4. *Let G be a graph on n vertices. If G contains a K_5 such that $N(K_5) \neq V(G)$ and each vertex of K_5 has degree at least $n-3$ in G, then $\chi_c(M(G)) = \chi(M(G))$.*

Proof. Suppose, to the contrary, that $\chi_c(M(G)) < \chi(M(G))$. Then $\chi_c(M(G)) = \frac{k}{d}$, where $d \geq 2$ and $\gcd(k,d) = 1$. As before, let $V(M(G)) = \{x_1, x_2, \ldots, x_n; x'_1, x'_2, \ldots, x'_n; u\}$, where $V = \{x_1, x_2, \ldots, x_n\}$ is the vertex-set of G, $V' = \{x'_1, x'_2, \ldots, x'_n\}$ is the set of the twins of vertices of V, and u is the root of $M(G)$. For $x \in V \cup V'$, denote by x' the twin of x; for $X \subseteq V \cup V'$, $X' = \{x' : x \in X\}$. As seen in (3.1), $M(G)$ has a (k,d)-partition $(X_0, X_1, \ldots, X_{k-1})$ such that, with $u \in X_0$ and

$$A = X_{d-1} \cup \ldots \cup X_1 \cup X_0 \cup X_{k-1} \cup \ldots \cup X_{k-d+1},$$

for any $x \in V \setminus A$,

$$\{x, x'\} \subseteq X_j \text{ for some } j,\ d \leq j \leq k-d, \tag{3.2}$$

and $X_i \neq \emptyset$, $0 \leq i \leq k-1$. Since each pair of the five vertices of the K_5 is adjacent, we may let $\{x_p, x_q, x_r, x_s, x_t\}$ be the five vertices of the K_5 such that $x_i \in X_i$, $i \in \{p,q,r,s,t\}$, and $0 \leq p < q < r < s < t \leq k-1$. By the given condition that $N(K_5) \neq V(G)$, there is

$$z \in V \setminus \left(\bigcup_{i \in \{p,q,r,s,t\}} N(x_i) \right),$$

and hence $d(x_i) \leq n-2$ in G for each $i \in \{p,q,r,s,t\}$.

Claim. *For any $i \in \{p,q,r,s,t\}$, if $d+1 \leq i \leq k-d-1$, then $d(x_i) = n-3$ in G, $X_i = \{x_i, x'_i\}$, $z' \in X_{i-1} \cup X_{i+1}$, and $|X_{i-1} \cap V'| = |X_{i+1} \cap V'| = 1$.*

Proof. Since $d+1 \leq i \leq k-d-1$, we have that $(X_{i-1} \cup X_{i+1}) \cap A = \emptyset$, and thus by (3.2), no vertex of X_{i-1} can be the twin of a vertex of X_{i+1}. If we let

$$R = X_{i-1} \cup X_{i+1} \cup X'_{i-1} \cup X'_{i+1},$$

then $|R \cap V| \geq 2$. We note that $x_i \in V$, and in $M(G)$, x_i is not adjacent to any vertex of $X_{i-1} \cup X_{i+1}$ and thus not to any vertex of R. Moreover, x_i is not adjacent to any vertex of $X_i \cup X'_i$. It follows that $d(x_i) \leq n-3$ in G with equality only if $z \in R$, $X_i = X'_i = \{x_i, x'_i\}$, and $|X_{i-1} \cap V'| = |X_{i+1} \cap V'| = 1$. By the given condition, equality $d(x_i) = n-3$ holds and therefore $z \in R$,

$X_i=\{x_i,x'_i\}$, and $|X_{i-1}\cap V'|=|X_{i+1}\cap V'|=1$. By (3.2), $z\in R$ implies that $z'\in X_{i-1}\cup X_{i+1}$. This proves the claim. ∎

Since each pair of the five vertices is adjacent, we have that

$$d\le p+d\le q,\quad q+d\le r,\quad r+d\le s,\quad \text{and}\quad s\le t-d\le k-d-1.$$

By the claim, for each $i\in\{r,s\}$, $d(x_i)=n-3$ in G, $X_i=\{x_i,x'_i\}$, $z'\in X_{i-1}\cup X_{i+1}$, and $|X_{i-1}\cap V'|=|X_{i+1}\cap V'|=1$. This means that $X_r=\{x_r,x'_r\}$, $X_s=\{x_s,x'_s\}$, $s=r+2$, and $X_{r+1}\cap V'=\{z'\}$. If $q\ge d+1$, then by the claim we have that $z'\in X_{q-1}\cup X_{q+1}$, which is impossible. Therefore, $q=d$, and thus $p=0$, that is, $x_p\in X_0$. So x_p is not adjacent to any vertex of $X_1\cup X_{k-1}$. Since the root $u\in X_0$ and u is adjacent to every vertex of V', we have that $X_1\cap V\ne\emptyset$ and $X_{k-1}\cap V\ne\emptyset$. On the other hand, it is given that $d(x_p)\ge n-3$ in G. Therefore, $d(x_p)=n-3$ and either $X_1=\{z\}$ or $X_{k-1}=\{z\}$. Without loss of generality, suppose that $X_1=\{z\}$. We note that $s=r+2$ implies $d=2$, and since $X_r=\{x_r,x'_r\}$ and $X_s=\{x_s,x'_s\}$, we may move z to X_{r+1} (note that $X_{r+1}\cap V'=\{z'\}$) to obtain a new (k,d)-partition $(Y_0,Y_1,\dots,Y_{k-1})$ with $Y_i=X_i$ if $i\notin\{1,r+1\}$, $Y_{r+1}=X_{r+1}\cup\{z\}$, and $Y_1=X_1\setminus\{z\}=\emptyset$, which contradicts Proposition 1.4 and proves Theorem 3.4. ∎

Recursively define $M^m(G)=M(M^{m-1}(G))$. It was conjectured [3,8] that If $m\le n-2$, then $\chi_c(M^m(K_n))=\chi(M^m(K_n))$. By Theorem 3.4, we conclude:

Corollary 3.5. *Let G be a graph on n vertices and K the set of vertices of degree $n-1$. If $|K|\ge 5$, then $\chi_c(M^2(G))=\chi(M^2(G))$.*

Proof. Let $H=M(G)$. Then H is a graph on $2n+1$ vertices, any five vertices of K induces a K_5 in H such that each vertex of the K_5 has degree $2(n-1)=|V(H)|-3$ in H. Since the root of $M(G)$ $(=H)$ is not in $N(K_5)$, applying Theorem 3.4 to H, we have that $\chi_c(M(H))=\chi(M(H))$, that is, $\chi_c(M^2(G))=\chi(M^2(G))$, as required. ∎

An immediate consequence of Corollary 3.5 is that if $n\ge 5$, then $\chi_c(M^2(K_n))=\chi(M^2(K_n))$, which is slightly weaker than the result of Chang, Huang, and Zhu [3] that if $n\ge 4$, then $\chi_c(M^2(K_n))=\chi(M^2(K_n))$. However, as pointed out by a referee, the case of $n=4$ can be handled by using Theorem 3.2 to exclude certain pairs of (k,d) values and show $\chi_c(M^2(K_4))=6$. In fact, by more involved arguments, we are able to weaken the condition of Theorem 3.4 by only requiring that G has a K_4 with the described property, which would include the case of $n=4$.

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