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Packing Cycles in Graphs

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A graph G is called *cycle Mengerian* (CM) if for all nonnegative integral function w defined on $V(G)$, the maximum number of cycles (repetition is allowed) in G such that each vertex v is used at most $w(v)$ times is equal to the minimum of $\sum\{w(x) : x \in X\}$, where the minimum is taken over all $X \subseteq V(G)$ such that deleting X from G results in a forest. The purpose of this paper is to characterize all CM graphs in terms of forbidden structures. As a corollary, we prove that if the fractional version of the above minimization problem always have an integral optimal solution, then the fractional version of the maximization problem will always have an integral optimal solution as well.

1. INTRODUCTION

Graphs considered in this paper are finite, simple, and undirected. For any vertex-set or edge-set Z of a graph G, we denote by $G\backslash Z$ the graph obtained from G by deleting Z; when Z is a singleton $\{z\}$, we may write $G\backslash z$ instead of $G\backslash \{z\}$.

Let $G = (V, E)$ be a graph with a nonnegative integral weight $w(v)$ on each $v \in V$. A collection $\mathscr{C}$ of cycles (repetition is allowed) of G is called a *cycle packing* if each vertex v of G is used most $w(v)$ times by members of $\mathscr{C}$; a set X of vertices in G is called a *feedback set* if $G\backslash X$ is a forest. Let $\nu_w(G)$ denote the maximum size of a cycle packing and let $\tau_w(G)$ denote the minimum total

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weight of a feedback set. Then it is not difficult to verify the following well-known inequality

$$\nu_w(G) \leqslant \tau_w(G). \tag{1.1}$$

In general, $\nu_w(G)$ and $\tau_w(G)$ do not have to be equal. As a matter of fact, the ratio of $\tau_w(G)$ over $\nu_w(G)$ can be arbitrarily large even when $w(v) = 1$ for all vertices $v \in V$, as shown by Erdös and Pósa [4]. Now a natural question is: when does (1.1) hold with equality? The purpose of this paper is to answer this question. Let us call *G cycle Mengerian* (CM) if $\nu_w(G) = \tau_w(G)$ for all nonnegative integral w. Our main result, Theorem 1.1, will characterize all CM graphs in terms of forbidden structures.

Other than purely mathematical curiosity, there is another motivation for studying CM graphs. The problems of computing $\nu_w(G)$ and $\tau_w(G)$ arise in a variety of applications and both problems are known to be NP-hard [5]. However, according to a powerful result of Grötschel *et al.* [6], in Case (1.1) which holds with equality for all nonnegative integral w, the problem of computing $\nu_w(G)$ and $\tau_w(G)$ is equivalent to finding a shortest cycle in a graph, which is solvable in polynomial time. Therefore, CM graphs form a class for which both $\nu_w(G)$ and $\tau_w(G)$ can be computed in polynomial time.

Before presenting our main theorem of the paper we need to define some graphs. A *Θ-graph* is a subdivision of $K_{2,3}$. A *wheel* is obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle. The new vertex is the *hub* of the wheel. A *W-graph* is a subdivision of a wheel. An *odd ring* (see Fig. 1) is a graph obtained from an odd cycle by replacing each edge $e = uv$ with *either* a triangle containing e or two triangles uab, vcd together with two additional edges ac and bd. A subdivision of an odd ring is called an *R-graph.*

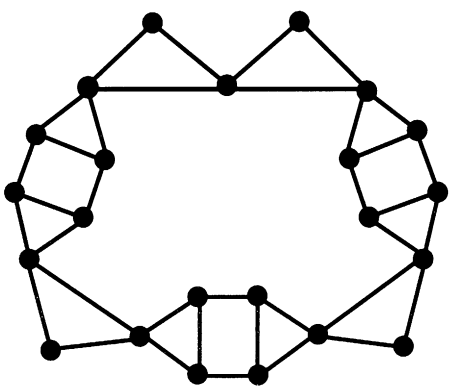

FIG. 1. An odd ring.

It will follow from Lemmas 4.1 and 4.2 that no CM graph can contain a Θ-graph, a W-graph, or an R-graph as an induced subgraph. Our main theorem, which is stated below, asserts that actually these graphs are the only obstructions to the desired minimax relation.

For convenience, we shall simply say that a graph G has a graph H if H is isomorphic to an *induced* subgraph of G.

THEOREM 1.1. *A graph is CM if and only if it has no Θ-graphs, nor W-graphs, nor R-graphs.*

Notice that a W-graph may have a Θ-graph or another W-graph. Similarly, an R-graph may also have a Θ-graph or a W-graph. We present Theorem 1.1 in the current form so as to make the statement cleaner. It is not difficult to determine (based on Theorem 1.1) all minimal non-CM and this is left to the reader.

One may also consider the edge version of the above cycle packing problem. Namely, to characterize all graphs G that have the following property: for all nonnegative integral function w defined on $E(G)$, the maximum number of cycles (repetition is allowed) in G such that each edge e is used at most $w(e)$ times is equal to the minimum of $\sum\{w(f): f \in F\}$, where the minimum is taken over all $F \subseteq E(G)$ such that $G \backslash F$ is a forest. This problem, however, is easy. When G is a graph for which every block is either K_2 or a cycle, it is easy to check that the minimax relation holds for all nonnegative integral w. On the other hand, when G is not such a graph, the minimax relation does not hold, for some nonnegative integral w, as explained below. Note that, under our assumption, G must have a subgraph H that is a subdivision of $K_4 \backslash e$. Let us define $w(f) = 1$ for all $f \in E(H)$ and $w(f) = 0$ for all other edges. Then the maximum equals one while the minimum equals two. Therefore, as we claimed, the minimax relation does not hold, under our choice of nonnegative integral w.

Theorem 1.1 has a very nice corollary concerning fractional cycle packings. Let $G = (V, E)$ be a graph with a nonnegative integral weight $w(v)$ on each vertex v. Define

$$v_w^*(G) = \max\left\{ \sum_{C\,:\,\text{cycle}} y(C) : \sum_{C\,:\,v \in V(C)} y_C \leqslant w(v),\ \forall v \in V; y \geqslant 0 \right\}. \quad (1.2)$$

Then G is called *cycle ideal* if $v_w^*(G) = \tau_w(G)$ for all nonnegative integral w. One may verify that inequality (1.1) can be refined as

$$v_w(G) \leqslant v_w^*(G) \leqslant \tau_w(G). \quad (1.3)$$

Therefore, every CM graph is cycle ideal. We shall justify, by virtue of Theorem 1.1, that the converse also holds.

COROLLARY 1.1. *A graph is cycle Mengerian if and only if it is cycle ideal.*

To prove Theorem 1.1, we introduce a partition property in Section 2, which is sufficient for a graph to be CM. We prove that, when piecing together graphs with this partition property, the resulting graph also has the property. In Section 3, we derive a structural theorem, which asserts that if a graph has no forbidden structures, then it can be expressed as "sums" of some prime graphs. In Section 4, we show that every prime graph enjoys the partition property, which, together with the results established in Section 2, yields our main theorem.

2. SUMS OF HYPERGRAPHS

As outlined in the last section, the basic idea underlying our proof of Theorem 1.1 is to express CM graphs as sums of some prime graphs. The purpose of this section is to derive results concerning summing operations. We shall state these results in terms of hypergraphs, since the more general form may have potential applications elsewhere and since the proofs are easier to describe in this way.

First, let us point out that, in this paper, the word *collection* actually means multiset. That is, if $X = \{x_1, x_2, \ldots, x_m\}$ is a collection, then it is possible that $x_i = x_j$ for some $i \neq j$. In contrast, in a *set* and in a *subset* (of a collection), all its members are distinct. The *size* $|X|$ of X is defined to be m. If $Y = \{y_1, y_2, \ldots, y_n\}$ is also a collection, then $X \cup Y$ is the collection $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$. Note that the size of the union of two collections is always the sum of the sizes of the two collections, which is different from what happens to the union of two sets.

A *hypergraph* is simply a collection Γ of subsets of a finite set V. Members of V and Γ are called *vertices* and *hyperedges*, respectively. For a nonnegative integral function w on V, a *w-matching* of Γ is a collection M of hyperedges such that each vertex x in V is used at most $w(x)$ times by members of M. A *transversal* of Γ is a minimal (under inclusion) set T of vertices such that $T \cap A \neq \emptyset$ for all members A of Γ. As usual, let $\nu_w(\Gamma)$ denote the maximum size of a w-matching of Γ, and let $\tau_w(\Gamma)$ denote the minimum of $\sum\{w(x) : x \in T\}$, where the minimum is taken over all transversals T of Γ. Since we do not consider a graph as a hypergraph, there is no conflict between the definitions of $\nu_w(\Gamma)$ and $\nu_w(G)$, or the definitions of $\tau_w(\Gamma)$ and $\tau_w(G)$. In fact, if Γ_G is the *cycle hypergraph* of G, that is, the hypergraph that consists of the vertex-sets of all cycles of G, then

$\nu_w(G) = \nu_w(\Gamma_G)$ and $\tau_w(G) = \tau_w(\Gamma_G)$. In addition, it is not difficult to see that (1.1) can be generalized to

$$\nu_w(\Gamma) \leqslant \tau_w(\Gamma) \tag{2.1}$$

for all Γ and w. Following [12], a hypergraph Γ is called *Mengerian* if (2.1) holds with equality for all nonnegative integral w. Under this terminology, we can see that a graph G is CM if and only if Γ_G is Mengerian. It is worth pointing out that a Mengerian hypergraph is also said to have the MFMC property in [2, 12] and the $\mathbf{Z}_+$-MFMC property in [10].

People familiar with integer programming may like the following equivalent definition of Mengerian hypergraphs. Let M be the hyperedge-vertex incidence matrix of Γ. Then Γ is Mengerian if and only if the linear system $\{Mx \geqslant e,\ x \geqslant 0\}$ is TDI [3, 11], where e is the all-one vector. This, by the Edmonds–Giles theorem [3, 11], amounts to that both of the following two problems:

$$\begin{array}{llcll} \max & y^T e & & \min & w^T x \\ \text{s.t.} & y^T M \leqslant w^T, & \text{and} & \text{s.t.} & Mx \geqslant e, \\ & y \geqslant 0 & & & x \geqslant 0, \end{array} \tag{2.2}$$

have integral optimal solutions, for all nonnegative integral vectors w. Now it is easy to see that what Corollary 1.1 claims is that, for a cycle hypergraph, if the minimum in (2.2) has an integral optimal solution for all nonnegative integral vectors w, then the maximum in (2.2) also has an integral optimal solution for all nonnegative integral vectors w.

Usually, it is very difficult to recognize Mengerian hypergraphs by using the definitions given above. In the following, we introduce a property, which is sufficient for a hypergraph to be Mengerian and is much easier to work with. Let Γ be a hypergraph with vertex set V. For any collection Λ of members of Γ, we shall let $d_\Lambda(x)$ denote the number of hyperedges in Λ that contain a vertex x in V. For any *subset* Λ of Γ, a *subpartition* of Λ consists of two collections Λ_1 and Λ_2 of members of Γ (which are not necessarily in Λ) such that

(i) $|\Lambda_1 \cup \Lambda_2| = |\Lambda|$,
(ii) $d_{\Lambda_1 \cup \Lambda_2}(x) \leqslant d_\Lambda(x)$ for all x in V, and
(iii) Each member of Λ with size 3 is contained in $\Lambda_1 \cup \Lambda_2$. (2.3)

We make two remarks on this notion: first, Λ_1 and Λ_2 are collections while Λ is a set; second, condition (iii) will play an important role in the proofs of

Theorems 2.3 and 2.4, although it looks not so natural. The subpartition (Λ_1, Λ_2) is called *equitable* at a vertex $x \in V$ if

$$\max\{d_{\Lambda_1}(x), d_{\Lambda_2}(x)\} \leqslant \lceil d_\Lambda(x)/2 \rceil. \tag{2.4}$$

(Λ_1, Λ_2) is *equitable* if it is equitable at all x in V. The hypergraph Γ is called *equitably subpartitionable* (ESP) if every subset Λ of Γ admits an equitable subpartition.

THEOREM 2.1. *Every ESP hypergraph is Mengerian.*

From our proof the reader is recommended to observe that Theorem 2.1 still holds even if (ii) and (iii) in (2.3) are dropped from the definition of a subpartition. As remarked earlier, we formulate our definition this way so that the ESP property will be preserved under the summing operations. To prove Theorem 2.1, we need the following result of Lovász [7, 8].

LOVÁSZ' THEOREM. *A hypergraph Γ is Mengerian if and only if $\nu_{2w}(\Gamma) \leqslant 2\nu_w(\Gamma)$ for all nonnegative integral functions w.*

Proof of Theorem 2.1. Let Γ be an ESP hypergraph on V. To show that Γ is Mengerian, by Lovász' theorem, we may turn to verify that $\nu_{2w}(\Gamma) \leqslant 2\nu_w(\Gamma)$ for all nonnegative integral functions w defined on V. We prove this by finding a w-matching of size at least $\nu_{2w}(\Gamma)/2$.

Let M be a $2w$-matching of Γ of size $\nu_{2w}(\Gamma)$ and, for each hyperedge A of Γ, let $M(A)$ be the number of times that A appears in M. Let Λ be the set of hyperedges A with $M(A)$ odd. Since Γ is ESP, Λ admits an equitable subpartition (Λ_1, Λ_2). Let M_0 be a collection of hyperedges such that each A appears $\lfloor M(A)/2 \rfloor$ times. It follows that $M = M_0 \cup M_0 \cup \Lambda$. For $i = 1, 2$, let $M_i = M_0 \cup \Lambda_i$. Then we claim that both M_1 and M_2 are w-matchings. To see this, let $i \in \{1, 2\}$ and let x be an arbitrary vertex of Γ. Since $2w(x) \geqslant d_M(x) = 2d_{M_0}(x) + d_\Lambda(x)$, we deduce from (2.4) that $w(x) - d_{M_0}(x) \geqslant d_{\Lambda_i}(x)$, which implies $w(x) \geqslant d_{M_0}(x) + d_{\Lambda_i}(x) = d_{M_i}(x)$, and thus the claim is proved. On the other hand, $|M_1| + |M_2| = 2|M_0| + |\Lambda_1| + |\Lambda_2|$, which, by (i) in (2.3), equals $2|M_0| + |\Lambda| = |M| = \nu_{2w}(\Gamma)$. Therefore, at least one of the two w-matchings M_1 and M_2 has a size of at least $\nu_{2w}(\Gamma)/2$. ∎

A graph G is called *ESP* if its cycle hypergraph Γ_G is ESP. The next corollary follows obviously from Theorem 2.1. We point out that all corollaries in this section can be deduced easily from the corresponding theorems by considering the cycle hypergraph.

COROLLARY 2.1. *Every ESP graph is CM.*

We shall repeatedly use the following definition in this paper. Let Γ_1 and Γ_2 be hypergraphs with vertex sets V_1 and V_2, respectively, such that

$V_1 \cap V_2 = \emptyset$. For $i = 1, 2$, let $x_{i1}, x_{i2}, \ldots, x_{ik}$ be k distinct vertices of Γ_i. A hypergraph Γ is said to be obtained from Γ_1 and Γ_2 by *identifying* $x_{11}, x_{12}, \ldots, x_{1k}$ with $x_{21}, x_{22}, \ldots, x_{2k}$ if its hyperedge set $\Gamma = \Gamma_1 \cup \Gamma_2$ and its vertex set $V = V_1 \cup V_2$, where each x_{1j} is made identical to x_{2j}, for $1 \leqslant j \leqslant k$. Identifying vertices in graphs can be defined analogously. The only difference is that, when multiple edges are created, we will delete one of them rather than keeping both.

The rest of this section is devoted to prove that ESP property is preserved under certain summing operations. First, we prove a couple of lemmas on subpartitions, which will be used several times in this section.

LEMMA 2.1. *Let Λ_1 and Λ_2 be two disjoint subsets of Γ. Suppose each Λ_i has a subpartition $(\Lambda_i^1, \Lambda_i^2)$. Then $(\Lambda_1^1 \cup \Lambda_2^j, \Lambda_1^2 \cup \Lambda_2^{3-j})$ is a subpartition of $\Lambda_1 \cup \Lambda_2$ for both $j = 1$ and 2.*

Proof. Note that $(\Lambda_1^1 \cup \Lambda_2^j) \cup (\Lambda_1^2 \cup \Lambda_2^{3-j}) = (\Lambda_1^1 \cup \Lambda_1^2) \cup (\Lambda_2^1 \cup \Lambda_2^2)$. Then the lemma follows from a straightforward verification of (2.3). ∎

LEMMA 2.2. *Let Γ be obtained from Γ_1 and Γ_2 by identifying $x_{11}, x_{12}, \ldots, x_{ik}$ with $x_{21}, x_{22}, \ldots, x_{2k}$. Let Λ_1 and Λ_2 be disjoint subsets of Γ for which $\Lambda_i \subseteq \Gamma_i$ $(i = 1, 2)$. Suppose each Λ_i has an equitable subpartition $(\Lambda_i^1, \Lambda_i^2)$ and suppose the subpartition $S = (\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ of $\Lambda_1 \cup \Lambda_2$ is not equitable. Then there exists an index j such that all the following hold.*

(i) *both $d_{\Lambda_1}(x_{1j})$ and $d_{\Lambda_2}(x_{2j})$ are odd*;

(ii) *there exists an index i such that $\lceil d_{\Lambda_1}(x_{1j})/2 \rceil = d_{\Lambda_1^i}(x_{1j})$ and $\lceil d_{\Lambda_2}(x_{2j})/2 \rceil = d_{\Lambda_2^{3-i}}(x_{2j})$; and*

(iii) $(d_{\Lambda_1^1}(x_{1j}) - d_{\Lambda_1^2}(x_{1j})) \cdot (d_{\Lambda_2^1}(x_{2j}) - d_{\Lambda_2^2}(x_{2j})) < 0$.

Proof. For $j = 1, 2, \ldots, k$, let x_j be the vertex obtained by identifying x_{1j} with x_{2j}. Let x be a vertex of Γ such that $x \neq x_j$ for all j. Then x is a vertex of some Γ_i. Observe that $d_{\Lambda_1^1 \cup \Lambda_2^2}(x) = d_{\Lambda_i^i}(x)$, $d_{\Lambda_1^2 \cup \Lambda_2^1}(x) = d_{\Lambda_i^{3-i}}(x)$, and $d_{\Lambda_1 \cup \Lambda_2}(x) = d_{\Lambda_i}(x)$. It follows that S is equitable at x. Therefore, there exists an index j such that S is not equitable at x_j. That is,

$$\max\{d_{\Lambda_1^1 \cup \Lambda_2^2}(x_j), d_{\Lambda_1^2 \cup \Lambda_2^1}(x_j)\} > \lceil d_{\Lambda_1 \cup \Lambda_2}(x_j)/2 \rceil. \tag{2.5}$$

Note that $d_{\Lambda_1 \cup \Lambda_2}(x_j) = d_{\Lambda_1}(x_j) + d_{\Lambda_2}(x_j)$, and, for $i = 1, 2$,

$$\lceil d_{\Lambda_1}(x_j)/2 \rceil + \lceil d_{\Lambda_2}(x_j)/2 \rceil \geqslant d_{\Lambda_1^i}(x_j) + d_{\Lambda_2^{3-i}}(x_j) = d_{\Lambda_1^i \cup \Lambda_2^{3-i}}(x_j). \tag{2.6}$$

Thus, by (2.5),

$$\lceil d_{\Lambda_1}(x_{1j})/2\rceil + \lceil d_{\Lambda_2}(x_{2j})/2\rceil > \lceil (d_{\Lambda_1}(x_{1j}) + d_{\Lambda_2}(x_{2j}))/2\rceil.$$

This inequality obviously implies that $d_{\Lambda_1}(x_{1j})$ and $d_{\Lambda_2}(x_{2j})$ are odd, which proves (i), and

$$\lceil d_{\Lambda_1}(x_{1j})/2\rceil + \lceil d_{\Lambda_2}(x_{2j})/2\rceil = \lceil (d_{\Lambda_1}(x_{1j}) + d_{\Lambda_2}(x_{2j}))/2\rceil + 1.$$

It follows that (2.6) must hold with equality for some i, and that proves (ii). Then, for this i, it is clear from (i) and (ii) above that $d_{\Lambda_1^i}(x_{1j}) > d_{\Lambda_1^{3-i}}(x_{1j})$ and $d_{\Lambda_2^{3-i}}(x_{1j}) > d_{\Lambda_1^i}(x_{1j})$, which proves (iii). ∎

THEOREM 2.2. *Suppose Γ is obtained by identifying k vertices of Γ_1 with k vertices of Γ_2 $(k = 0, 1)$. If both Γ_1 and Γ_2 are ESP, then so is Γ.*

Proof. Let Λ be a subset of Γ, let $\Lambda_1 = \Lambda \cap \Gamma_1$, and let $\Lambda_2 = \Lambda - \Lambda_1$. Since each Γ_i is ESP, Λ_i has an equitable subpartition $(\Lambda_i^1, \Lambda_i^2)$. When $k = 0$, by Lemmas 2.1 and 2.2, we conclude immediately that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ. When $k = 1$, let x be the common vertex of Γ_1 and Γ_2. Without loss of generality, let $d_{\Lambda_i^1}(x) \leqslant d_{\Lambda_i^2}(x)$ for $i = 1, 2$. Then it follows from Lemmas 2.1 and 2.2(iii) that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ. ∎

Let G_1 and G_2 be two graphs. The 0-*sum* of G_1 and G_2 is obtained by taking the disjoint union of G_1 and G_2; the 1-*sum* is obtained by identifying a vertex of G_1 with a vertex of G_2. The following corollary follows instantly from Theorem 2.2.

COROLLARY 2.2. *Suppose G is the* 0- *or* 1-*sum of G_1* and G_2. *If both G_1 and G_2 are ESP, then so is G.*

THEOREM 2.3. *Let Γ be obtained by identifying vertices x_{11}, x_{12} of Γ_1 with vertices x_{21}, x_{22} of Γ_2. For $i = 1, 2$, let Γ_i' be obtained from Γ_i by adding a new vertex z_i and a new edge $\{x_{i1}, x_{i2}, z_i\}$. If both Γ_1' and Γ_2' are ESP, then so is Γ.*

Proof. Let Λ be a set of hyperedges of Γ. We need to find an equitable subpartition of Λ. Set $\Lambda_1 = \Lambda \cap \Gamma_1$ and $\Lambda_2 = \Lambda - \Lambda_1$. We consider the following two cases.

Case 1. At least one of $d_{\Lambda_i}(x_{ij})$, for $i, j \in \{1, 2\}$, is even. Let us assume that $d_{\Lambda_1}(x_{11})$ is even. Since each Γ_i' is ESP, Λ_i has an equitable subpartition $(\Lambda_i^1, \Lambda_i^2)$. Note that z_i is not contained in any member of Λ_i, so all members of $\Lambda_i^1 \cup \Lambda_i^2$ are members of Γ_i. Without loss of generality, for $i = 1, 2$, let us assume that $d_{\Lambda_i^1}(x_{i2}) \leqslant d_{\Lambda_i^2}(x_{i2})$. Then, we conclude from Lemmas 2.1

and 2.2(i) and (iii) that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ.

Case 2. $d_{\Lambda_i}(x_{ij})$ is odd, for all $i, j \in \{1,2\}$. Let $A_i = \{x_{i1}, x_{i2}, z_i\}$. Since each Γ_i' is ESP, $\Lambda_i \cup \{A_i\}$ admits an equitable subpartition. As $d_{\Lambda_i \cup \{A_i\}}(z_i) = 1$ and $|A_i| = 3$, from conditions (ii) and (iii) of (2.3) we conclude that A_i appears precisely once in this subpartition. Let $(\Lambda_i^1 \cup \{A_i\}, \Lambda_i^2)$ denote this subpartition. Since A_i is the only hyperedge containing z_i in Γ_i', all members of $\Lambda_i^1 \cup \Lambda_i^2$ are in Γ_i. In fact, it is not difficult to verify that $(\Lambda_i^1, \Lambda_i^2)$ is an equitable subpartition of Λ_i. Furthermore, for all $i, j \in \{1,2\}$, we have $d_{\Lambda_i^1 \cup \{A_i\}}(x_{ij}) \leqslant \lceil d_{\Lambda_i \cup \{A_i\}}(x_{ij})/2 \rceil$, which can be simplified as $d_{\Lambda_i^1}(x_{ij}) < d_{\Lambda_i}(x_{ij})/2$. Then, we conclude from Lemmas 2.1 and 2.2(ii) that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ. ∎

A 2-*sum* of two graphs G_1 and G_2 is obtained by first choosing a triangle $x_i y_i z_i$ from G_i $(i = 1,2)$ such that z_i has degree two in G_i, then deleting z_i from G_i $(i = 1,2)$, and finally, identifying $x_1 y_1$ with $x_2 y_2$. The difference between our 2-sum and the ordinary 2-sum is the requirement on the extra vertices z_i. The necessity of such a requirement in our definition, as well as in Theorem 2.3, can be understood from the structure of an odd ring: Let C be an odd cycle, let $e = xy$ be an edge on C, and let G be an odd ring obtained from C in which e is replaced by a triangle xyz. Then both $G \backslash z$ and the triangle xyz are ESP, whereas their combination, G, is not.

COROLLARY 2.3. *Suppose G is a 2-sum of G_1 and G_2. If both G_1 and G_2 are ESP, then so is G.*

With the same spirit as that of Theorem 2.3, we have the following more complex result. Since its proof is very similar to the proof of Theorem 2.3, some details will be omitted.

THEOREM 2.4. *Let $B_i = \{x_{i1}, x_{i2}, x_{i3}\}$ $(i = 1,2)$ be an edge of Γ_i and let Γ be obtained by identifying x_{1j} with x_{2j} $(j = 1,2,3)$. For $i = 1,2$ and $1 \leqslant j < k \leqslant 3$, let Γ_{ijk} be obtained from Γ_i by adding a new vertex x_{ijk} and a new edge $A_{ijk} = \{x_{ijk}, x_{ij}, x_{ik}\}$. If all Γ_{ijk} are ESP, then so is Γ.*

Proof. Let Λ be a set of hyperedges of Γ. We need to find an equitable subpartition of Λ. Set $\Lambda_1 = \Lambda \cap \Gamma_1$ and $\Lambda_2 = \Lambda - \Lambda_1$. We consider the following three cases.

Case 1. At least two of the three sets in $\{\{d_{\Lambda_1}(x_{1j}), d_{\Lambda_2}(x_{2j})\} : j = 1,2,3\}$ contain even numbers. This is an analogue of Case 1 in the proof Theorem 2.3. Let us assume that $\{d_{\Lambda_1}(x_{11}), d_{\Lambda_2}(x_{21})\}$ and $\{d_{\Lambda_1}(x_{12}), d_{\Lambda_2}(x_{22})\}$ contain even numbers. Since each Γ_{i12} is ESP, Λ_i has an equitable subpartition

$(\Lambda_i^1, \Lambda_i^2)$. As before, let us assume that $d_{\Lambda_i^1}(x_{i3}) \leqslant d_{\Lambda_i^2}(x_{i3})$, for $i = 1, 2$. Then $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ.

Case 2. Exactly one of the three sets in $\{\{d_{\Lambda_1}(x_{1j}), d_{\Lambda_2}(x_{2j})\} : j = 1, 2, 3\}$, say when $j = 3$, contains even numbers. This is an analogue of Case 2 in the proof Theorem 2.3. Since each Γ_{i12} is ESP, $\Lambda_i \cup \{A_{i12}\}$ has an equitable subpartition, which can be expressed as $(\Lambda_i^1 \cup \{A_{i12}\}, \Lambda_i^2)$ and such that $(\Lambda_i^1, \Lambda_i^2)$ is an equitable subpartition of Λ_i. Then the same argument as we used in the proof of Theorem 2.3 shows that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ.

Case 3. $d_{\Lambda_i}(x_{ij})$ is odd for all i and j. If $B_1 \in \Lambda_1$, then $B_2 \notin \Lambda_2$ by the definition of Λ_i $(i = 1, 2)$. In this case we can replace the pair Λ_1, Λ_2 by $\Lambda_1 - \{B_1\}, \Lambda_2 \cup \{B_2\}$, and the result follows from the proof in Case 1. Therefore, we may assume that $B_i \notin \Lambda_i$ for $i = 1, 2$. Since each Γ_{i12} is ESP, $\Lambda_i \cup \{B_i\}$ has an equitable subpartition $(\Lambda_i^1 \cup \{B_i\}, \Lambda_i^2)$. Again, it is routine to verify that $(\Lambda_i^1, \Lambda_i^2)$ is an equitable subpartition of Λ_i. In addition, for all i and j, we have $d_{\Lambda_i^1 \cup \{B_i\}}(x_{ij}) \leqslant \lceil d_{\Lambda_i \cup \{B_i\}}(x_{ij})/2 \rceil$, which can be simplified as $d_{\Lambda_i^1}(x_{ij}) < d_{\Lambda_i}(x_{ij})/2$. Thus, we conclude from Lemmas 2.1 and 2.2(ii) that $(\Lambda_1^1 \cup \Lambda_2^2, \Lambda_1^2 \cup \Lambda_2^1)$ is an equitable subpartition of Λ. ∎

A triangle T of a graph G is called *stable* if $G \backslash V(T)$ is connected and every vertex of T has degree at least three in G. A 3-*sum* of two graphs G_1 and G_2 is obtained by identifying a stable triangle of G_1 with a stable triangle of G_2. A careful reader may have noticed that the requirement on the triangles in the definition of a 3-sum seems more than what we need to carry Theorem 2.4 from hypergraphs to graphs. But in fact, one can see from our analysis in the next two sections that if some of these triangles are not stable, then either the 3-sum is also a 0-, 1- or 2-sum of two smaller graphs, or some G_{ijk} in the Corollary 2.4 is not ESP. Thus, requiring the triangles to be stable is the right way to define 3-sum.

COROLLARY 2.4. *Let G be a* 3-*sum of G_1 and G_2 over a triangle $x_1x_2x_3$. For $i = 1, 2$ and $1 \leqslant j < k \leqslant 3$, let G_{ijk} be obtained from G_i by adding a new vertex x_{ijk} and two new edges $x_{ijk}x_j$ and $x_{ijk}x_k$. If all G_{ijk} are ESP, then so is G.*

3. A DECOMPOSITION OF CM GRAPHS

Let a Δ-*graph* be obtained from a triangle xyz by adding three internally vertex disjoint paths, one from x to y, one from y to z, and one from z to x. Note that a Δ-graph is a special R-graph. A *rooted graph* consists of a graph G and a specified set F of edges such that each $f \in F$ belongs to a triangle and each triangle in G contains at most one edge from F. By *adding pendent* *triangles* to the rooted graph G we mean the following operation: to each edge $f = xy$ in F, we introduce a new vertex z_f and two new edges xz_f and yz_f. The following is the main result of this section.

THEOREM 3.1. *For any graph G, at least one of the following holds*:

(i) *G is a k-sum of two smaller graphs, for some $k = 0, 1, 2, 3$;*
(ii) *G has a Θ-graph, a W-graph, or a Δ-graph*;
(iii) *G is obtained from a rooted 2-connected line graph by adding pendent triangles.*

We break the proof of this result into a sequence of lemmas. Our proof heavily relies on two induced subgraphs called a diamond and a claw, respectively. We aim to prove in Lemma 3.6 that if neither of (i) and (ii) occurs, then in the presence of a diamond or a claw, G contains a separating triangle T such that $G \backslash V(T)$ has a component consisting of a single vertex x of degree two in G. Assuming this statement, let X be the set of these vertices x. Then we prove that $G_X = G \backslash X$ is a rooted graph with the set of root edges $F = \{a_x b_x : x \in X\}$, where a_x and b_x are the two neighbors of each $x \in X$. We will also prove that G_X is 2-connected and has neither a diamond nor a claw. Finally, we deduce from Beineke's theorem that G_X is a line graph and thus (iii) holds.

A path with end-vertices x and y is called an *xy-path*.

LEMMA 3.1. *Let H be a subdivision of K_4 and let a and b be two of the four degree-three vertices. Let G be obtained from H by adding edges such that all these edges are incident with either a or b. Then G has a W-graph.*

Proof. Let c and d be the other two vertices of H of degree three. For distinct vertices x and y in $\{a, b, c, d\}$, let P_{xy} denote the path of H obtained by subdividing the edge xy of K_4. By deleting vertices, if necessary, we may assume that each P_{xy} is an induced path of G. It follows that edges of G that are not in any of these paths must be between $\{a, b\}$ and $V(P_{cd}) - \{c, d\}$. If all these edges are incident with only one of a and b, say a, then it is easy to see that G itself is a W-graph, with a as the hub. Consequently, we may assume that both a and b have neighbors in $V(P_{cd}) - \{c, d\}$. For each x in $V(P_{cd})$, let P_{cx} be the unique cx-path of P_{cd}. We choose a vertex x in $V(P_{cd})$, with $V(P_{cx})$ minimal, such that $V(P_{cx}) - \{c\}$ contains both neighbors of a and b. Clearly, at least one of a and b, say b, has no neighbors in $V(P_{cx}) - \{c, x\}$. Then it is easy to see that $V(P_{ac}) \cup V(P_{bc}) \cup V(P_{ab}) \cup V(P_{cx})$ induces a W-graph in G, with a as the hub. ∎

LEMMA 3.2. *Let G be a graph with at least six vertices and let xy be an edge of G such that $G \backslash \{x, y\}$ is disconnected. Then G is a 2-sum of two smaller* *graphs over xy, unless $G\backslash\{x, y\}$ has only two components with one being an isolated vertex.*

Proof. If all components of $G\backslash\{x, y\}$ are isolated vertices, let G_1' consist of two of these vertices. If $G\backslash\{x, y\}$ has a component with two or more vertices, let G_1' be such a component. Let $G_2' = G\backslash(V(G_1') \cup \{x, y\})$. Suppose we are not in the situation that $G\backslash\{x, y\}$ has only two components with one being an isolated vertex. Then it is clear that each G_i' has at least two vertices. For $i = 1, 2$, let G_i be obtained from $G\backslash V(G_i')$ by adding a new vertex z_i and two new edges $z_i x$ and $z_i y$. It follows that G is a 2-sum of G_1 and G_2 with both being smaller than G. ∎

A *diamond* is a graph obtained from K_4 by deleting an edge. The following is a corollary of the last two lemmas.

LEMMA 3.3. *If a graph G has a diamond D, then at least one of the following holds*:

(i) *D has a vertex of degree two in G*;
(ii) *G can be expressed as a 2-sum of two smaller graphs, and the two triangles of D are contained in different parts*;
(iii) *G has a W-graph.*

Proof. Let $V(D) = \{a, b, c, d\}$ and let a and b be the two vertices of degree three in D. If c and d are contained in the same component of $G\backslash\{a, b\}$, then $G\backslash\{a, b\}$ has an induced cd-path P. By applying Lemma 3.1 to $D \cup P$ we deduce that (iii) holds. Therefore, we may assume that c and d are contained in different components G_c and G_d of $G\backslash\{a, b\}$. It is clear that (i) holds if G_c or G_d consists of only one vertex. On the other hand, if both G_c and G_d contain two or more vertices, then (ii) holds since its first half follows from Lemma 3.2 and its second half follows from the proof of Lemma 3.2. ∎

An edge $e = xy$ is called a *chord* of a cycle C if $e \notin E(C)$ yet both x and y are in $V(C)$. A *Θ_1-graph* is obtained from a cycle of length at least six by adding precisely one chord such that no triangle is created. A *Θ_2-graph* is obtained from a cycle of length at least six by adding precisely two chords xy and xz such that yz is an edge of the cycle; we shall call xyz the *inscribed triangle* of the Θ_2-graph.

LEMMA 3.4. *If a graph G has a Θ_1-graph H with chord e, then at least one of the following holds*:

(i) *G has a Θ_2-graph whose inscribed triangle contains e*;

(ii) *G can be expressed as a 2-sum of two smaller graphs over e;*
(iii) *G has a W-graph.*

Proof. Let $a, b \in V(H)$ be the two ends of e and let P_1 and P_2 be the two components of $H \setminus \{a, b\}$. If P_1 and P_2 are contained in different components of $G \setminus \{a, b\}$, then we deduce from Lemma 3.2 that (ii) holds. Next, we consider the case when $G \setminus \{a, b\}$ has a component that contains both paths P_1 and P_2. In this component, we choose the shortest path P between P_1 and P_2. Then P is an induced path. Let $x_0, x_1, \ldots, x_p, x_{p+1}$ be the vertices of P such that $x_0 \in V(P_1)$, $x_{p+1} \in V(P_2)$, and they are ordered as in P. From the minimality of P, no x_i $(i > 1)$ has a neighbor in P_1 and no x_i $(i < p)$ has a neighbor in P_2. Let us now distinguish among three cases.

Case 1. x_1 has three or more neighbors in P_1. In this case, $V(P_1) \cup \{a, b, x_1\}$ induces a W-graph, with x_1 as the hub, and thus (iii) holds. By symmetry, (iii) also holds if x_p has three or more neighbors in P_2.

Case 2. x_1 has precisely one neighbor in P_1 and x_p has precisely one neighbor in P_2. In this case, let F denote the K_4 subdivision consisting of H and P. Then we deduce from Lemma 3.1 that (iii) holds, where a, b in F correspond to the vertices a, b in Lemma 3.1.

Case 3. If none of the previous cases happens, then, by symmetry, we may assume that x_1 has precisely two neighbors in P_1. Clearly (iii) holds if no x_i $(1 \leqslant i \leqslant p)$ is adjacent to any of a and b. So we can choose the smallest i in $\{1, 2, \ldots, p\}$ such that x_i is adjacent to a or b, say a. If x_i is not adjacent to b, then (iii) holds since $V(P_1) \cup \{a, b, x_1, \ldots, x_i\}$ induces a subdivision of K_4. Thus we can assume that x_i is also adjacent to b. If i is 1 or p, then it is easy to see that (iii) holds again. If $1 < i < p$, then (i) holds since the subgraph induced by $V(P_1') \cup V(P_2) \cup \{a, b, x_1, \ldots, x_i\}$ is a Θ_2-graph with inscribed triangle abx_i, where P_1' is the part of P_1 from x_0 to $\{a, b\}$ that avoids the other neighbor of x_1 in P_1. ∎

If T is a triangle of a graph G for which $G \setminus V(T)$ has more components than G, then T is called a *separating* triangle.

LEMMA 3.5. *If a graph G has a Θ_2-graph H with inscribed triangle xyz, then either G has a W-graph, or xyz is a separating triangle of G such that $H \setminus \{x, y, z\}$ is not entirely contained in any component of $G \setminus \{x, y, z\}$.*

Proof. By renaming the three vertices x, y, and z, if necessary, we may assume that xy and xz are the two chords of the cycle $H \setminus \{xy, xz\}$. Let P_1 and P_2 be the two paths of $H \setminus \{x, y, z\}$ such that y is adjacent to an end of P_1. Suppose $H \setminus \{x, y, z\}$ is contained in a component of $G \setminus \{x, y, z\}$. Then $G \setminus \{x, y, z\}$ has a path P between P_1 and P_2. Let us choose P as short as possible. It follows that P is an induced path. Let $x_0, x_1, \ldots, x_p, x_{p+1}$ be the vertices of P such that $x_0 \in V(P_1)$, $x_{p+1} \in V(P_2)$, and they are ordered as in P. From the minimality of P, no x_i $(i > 1)$ has a neighbor in P_1 and no x_i $(i < p)$ has a neighbor in P_2. We now prove that G has a W-graph.

Case 1. x_1 has three or more neighbors in P_1. In this case, $V(P_1) \cup \{x, y, x_1\}$ induces a W-graph, with x_1 as the hub. By symmetry, G also has a W-graph if x_p has three or more neighbors in P_2.

Case 2. x_1 has precisely one neighbor in P_1 and x_p has precisely one neighbor in P_2. If no x_i $(1 \leqslant i \leqslant p)$ is adjacent to any of y and z, then $V(H) \cup V(P)$ induces a W-graph, with x as the hub. On the other hand, if some x_i $(1 \leqslant i \leqslant p)$ is adjacent to y or z, say z, let i denote the smallest subscript for which $x_i z$ is an edge and let F denote the subgraph induced by $V(P_1) \cup \{x, y, z, x_1, \ldots, x_i\}$. It is easy to see that F consists of a subdivision of K_4 and possibly some edges incident with x or y. By Lemma 3.1, F, and thus G, must have a W-graph.

Case 3. If none of the previous cases occurs, then, by symmetry, we may assume that x_1 has precisely two neighbors in P_1. Since the subgraph induced by $V(P_2) \cup \{x, y, x_1, \ldots, x_p\}$ is connected, it contains a path P' from x_1 to $\{x, y\}$ such that P' is the shortest among all such paths in this subgraph. Let $u \in V(P')$ be the neighbor of x or y. Since all internal vertices of P' are in $V(P_2) \cup \{x_2, \ldots, x_p\}$ and no vertex in this set is adjacent to any vertex of P_1, we may assume that u is adjacent to both x and y, because otherwise, by the minimality of P', the set $V(P_1) \cup V(P') \cup \{x, y\}$ induces a subdivision of K_4. Clearly, u is not in $V(P_2)$ and thus $u = x_i$ for some i. Let us assume that u is not adjacent to z, because otherwise $\{x, y, z, u\}$ induces a K_4. Therefore, $\{x, y, z, u\}$ induces a diamond. Observe that both z and u have degree at least three and there is a zu-path in $V(P_2) \cup \{z, x_i, x_{i+1}, \ldots, x_p\}$, which avoids $\{x, y\}$. Thus we conclude from Lemma 3.3 that G has a W-graph. ∎

LEMMA 3.6. *If a* 2-*connected graph G has a separating triangle T*, *then at least one of the following holds*:

(i) *G is a k-sum of two smaller graphs*, *for some* $k = 2, 3$;
(ii) *G has a* Δ-*graph or a* W-*graph*;
(iii) $G \setminus V(T)$ *has precisely two components with one of them being an isolated vertex of degree two in G*.

Proof. Let $V(T) = \{x_1, x_2, x_3\}$ and let us assume that (i) does not hold. We need to show that either (ii) or (iii) holds. We first consider the case when $G \setminus V(T)$ has exactly two components G_1 and G_2. Since G is not a 3-sum of two other graphs, T is not stable in some $G \setminus V(G_i)$, say $i = 2$. Thus some x_j, say $j = 1$, has degree two in $G \setminus V(G_2)$. It follows that G_1 is a component of $G \setminus \{x_2, x_3\}$. But G is not a 2-sum of two smaller graphs, we conclude from Lemma 3.2 that G_1 is an isolated vertex of degree two and thus (iii) holds.

Next, we consider the case when $G\backslash V(T)$ has more than two components. For each component H of $G\backslash V(T)$, let $T(H)$ be the set of vertices in T that have neighbors in H. Choose any three distinct components G_1, G_2, and G_3 of $G\backslash V(T)$. Since G is 2-connected, $|T(G_i)|\geqslant 2$ for all i. From Lemma 3.2 we deduce that, for each $i\neq j$, $G\backslash\{x_i,x_j\}$ has at most two components. It follows that each x_i belongs to at least two of the three sets in $\{T(G_j): j = 1,2,3\}$. Therefore, we can rename the vertices of T, if necessary, such that $T(G_1)\supseteq\{x_2,x_3\}$, $T(G_2)\supseteq\{x_1,x_3\}$, and $T(G_3)\supseteq\{x_1,x_2\}$. Let i,j,k be a permutation of 1, 2, 3. It is clear that we can find an x_jx_k-path P_{jk} such that the path has at least one interior vertex and all its interior vertices are in G_i. Let us choose such P_{jk} as short as possible. Then $V(P_{jk})$ induces a cycle. Now it is easy to see that (ii) holds since either the subgraph induced by $V(P_{12}\cup P_{23}\cup P_{31})$ is a Δ-graph or some x_i, say x_1, has a neighbor in $P_{23}\backslash\{x_2,x_3\}$, which implies that the subgraph induced by $V(P_{23})\cup\{x_1\}$ is a W-graph, with x_1 as the hub. ∎

LEMMA 3.7. *Let X be a set of degree-two vertices in a 2-connected graph G such that each vertex in X is in a triangle. If $G\backslash X$ is a 2- or 3-sum of two smaller graphs, then either G is a 2- or 3-sum of two smaller graph, or G has a Δ- or W-graph.*

Proof. If X contains two adjacent vertices, then it is easy to see that $G = K_3$ and thus the result holds trivially. Therefore, we may assume that no two vertices in X are adjacent. Suppose $G\backslash X$ is a 2-sum of two smaller graphs G_1' and G_2'. It is clear that X can be partitioned into X_1 and X_2 such that for each x in X_i ($i = 1,2$), the two neighbors of x are both in G_i'. For $i = 1,2$, let G_i be obtained from G_i' by putting the vertices in X_i back. Then it is easy to verify that G is a 2-sum of G_1 and G_2, both are smaller than G. Next, suppose $G\backslash X$ is a 3-sum of two smaller graphs over a triangle T. If each x in X has at most one neighbor in T, then, similar to the previous case, G is a 3-sum of two smaller graphs. If some x in X has both neighbors in T, then $G\backslash V(T)$ has three or more components. It follows from Lemma 3.6 that either G is a 2- or 3-sum of two smaller graphs, or G has a Δ- or W-graph. ∎

To state the next lemma, we need to define several more graphs. F_5 is obtained from a path on five vertices by adding a new vertex and making it adjacent to all vertices in the path. F_5^+ is obtained from F_5 by adding an edge between the two nonadjacent vertices of degree three. A *Θ_3-graph* is obtained from $K_{2,3}$ by subdividing an edge an arbitrary number of times (possibly not at all) and then adding an edge between the two vertices of degree three. A *K_4'-graph* is obtained from K_4 by subdividing at least two of the three edges from some star. A *K_4^+-graph* is obtained from a K_4 with vertex set $\{a, b, c, d\}$ by subdividing ab at least once and then adding a new vertex x and two new edges xa and xc. We shall call an induced $K_{1,3}$ a *claw*.

LEMMA 3.8. *A 2-connected graph has a claw if and only if it has an induced subgraph that is isomorphic to* F_5, F_5^+, *a* Θ*-graph, a* Θ_i*-graph* $(i = 1, 2, 3)$, *a* K_4'*-graph, or a* K_4^+*-graph.*

Proof. The "if" part is obvious since all the listed graphs have claws. To prove the "only if" part, let G be a 2-connected graph with a claw. Clearly, we may assume that G is *minimal* with this property, that is, every proper induced subgraph of G is either not 2-connected or claw-free. In particular, for every vertex z of G, every block of $G \backslash z$ is claw-free. Let edges xa_1, xa_2, and xa_3 form a claw. Then, we deduce that, for each vertex $z \notin \{x, a_1, a_2, a_3\}$, some xa_i and xa_j are contained in different blocks of $G \backslash z$. It follows that x is a cut-vertex of $G \backslash z$ that separates some a_i from some other a_j. Equivalently, for every vertex $z \notin \{x, a_1, a_2, a_3\}$, the set $\{x, z\}$ is a vertex-cut of G that separates some a_i from some other a_j.

Since G is 2-connected and x has degree of at least three, there must exist a vertex y other than x such that the degree of y is at least 3. Now the 2-connectivity of G guarantees the existence of a path in G from a_i to y which avoids x; by taking an appropriate section of this path, we see that for $i = 1, 2, 3$, there is a vertex $b_i \neq x$ and a path P_i from x to b_i such that a_i is in P_i, b_i has degree at least three, and all interior vertices of P_i have degree two in G (possibly $b_i = a_i$). We claim that $V(P_1 \cup P_2 \cup P_3) = V(G)$. Suppose, on the contrary, that some vertex z of G is not in $V(P_1 \cup P_2 \cup P_3)$. Then, without loss of generality, we may assume that a_1 and a_2 are separated from a_3 by $\{x, z\}$. Let G_3 be the component of $G \backslash \{x, z\}$ that contains a_3. Then b_3 is also contained in G_3. Let G_3' be the subgraph of G induced by $V(G_3) \cup \{x, z\}$. Since G is 2-connected, G_3' must have an xz-path P with at least one interior vertex. Choose such a path P as short as possible. Then P is an induced path, except for a possible edge xz. It follows that $V(G_3') - V(P) \neq \emptyset$, since otherwise all vertices in G_3 would be on the interior of P and hence have degree at most two in G_3', contradicting the fact that b_3, a vertex of G_3, is of degree at least three in G_3'. Therefore, $G \backslash (V(G_3') - V(P))$ is a proper induced subgraph and thus it should be either claw-free or not 2-connected. However, this graph has a claw $\{xa_1, xa_2, xa\}$, where a is the neighbor of x in P, and it is also 2-connected since it is obtained from a 2-connected graph G by replacing a part of a 2-separation with a path. This contradiction completes the proof of our claim.

Depending on the relationship between b_i's, we distinguish among the following three cases.

Case 1. $b_1 = b_2 = b_3$. If xb_1 is not an edge, then G is a Θ-graph. If $a_i = b_i$ for some i, then G is a Θ_1-graph. Next, we assume that xb_1 is an edge and

$a_i \neq b_i$ for all i. Without loss of generality, let us also assume that $|V(P_1)| \geqslant |V(P_2)| \geqslant |V(P_3)|$. Then a_2 is the only interior vertex of P_2 because otherwise $G \backslash a_3$ would have a Θ_1-graph, which contradicts the minimality of G. Therefore, a_3 is also the only interior vertex of P_3 and thus G is a Θ_3-graph.

Case 2. $b_1 = b_2 \neq b_3$. In this case, the above claim implies that $a_3 \neq b_3$, and b_3 must be adjacent to b_1 and x. It follows that $b_i = a_i$ for some $i = 1, 2$, for otherwise, $G \backslash a_3$ would have a Θ-graph, contradicting the minimality of G. Therefore, G is a Θ_2-graph with inscribed triangle xa_ib_3.

Case 3. b_1, b_2 and b_3 are all distinct. In this case, by the above claim each b_i is adjacent to at least one other b_j. It follows that there are at least two edges between b_1, b_2 and b_3. We first consider the case when some two of b_1, b_2 and b_3, say b_1 and b_3, are not adjacent. Since each b_i has degree at least three, for $i = 1, 3$, xb_i must be an edge not in P_i. From the minimality of G we deduce that, for $i = 1, 3$, a_i is the only interior vertex of P_i, because otherwise, $\{x, a_i, b_i, b_{4-i}\}$ would induce a claw that is contained in a block of $G \backslash a_{4-i}$. In addition, we must have $a_2 = b_2$ because otherwise $G \backslash \{a_1, a_3\}$ would be a Θ-graph. Consequently, $G = F_5$. Next, we assume that $b_1b_2b_3$ is a triangle. If there are no other edges, then G is a K_4'-graph. Thus, we may assume that xb_i is an edge not in P_i for some i, say $i = 1$. From the minimality of G we deduce that $a_i = b_i$ for some $i \neq 1$, say $i = 2$, because otherwise, $\{x, b_1, a_2, a_3\}$ would induce a claw that is contained in a block of $G \backslash a_1$. It follows that $a_3 \neq b_3$, as $\{x, a_1, a_2, a_3\}$ should induce a claw. Also from the minimality of G we deduce that a_1 is the only interior vertex of P_1, because otherwise $\{x, a_1, b_1, a_3\}$ would induce a claw in $G \backslash b_2$, which is a block. Now, it is straightforward to verify that G is either F_5^+ when xb_3 is an edge, or a K_4^+-graph when xb_3 is not an edge. ∎

We also need the following characterization of line graphs [1].

BEINEKE'S THEOREM. *A graph is a line graph if and only if it does not have any of the nine graphs below as an induced subgraph* (Fig. 2).

Proof of Theorem 3.1. Let G be a graph for which neither (i) nor (ii) holds. We need to show that (iii) must hold. Clearly, G is 2-connected. Let us also assume that G is not a line graph.

We first consider the case when G has at most five vertices. Since G is not a line graph and it does not have W-graphs, we conclude from Beineke's Theorem that G has a claw. Then, since G has no Θ-graphs, we deduce from Lemma 3.8 that G is a Θ_3-graph. Clearly, (iii) holds in this case.

Next, we assume that G has at least six vertices. Let X be the set of vertices x for which there is a separating triangle T_x such that x is a component of $G \backslash V(T_x)$. Since G is 2-connected and has no W-graphs, each $x \in X$ must have degree two. Let a_x, b_x, and c_x be the three vertices of T_x, with a_x and b_x being the neighbors of x. Then both a_x and b_x have degree greater than two, which

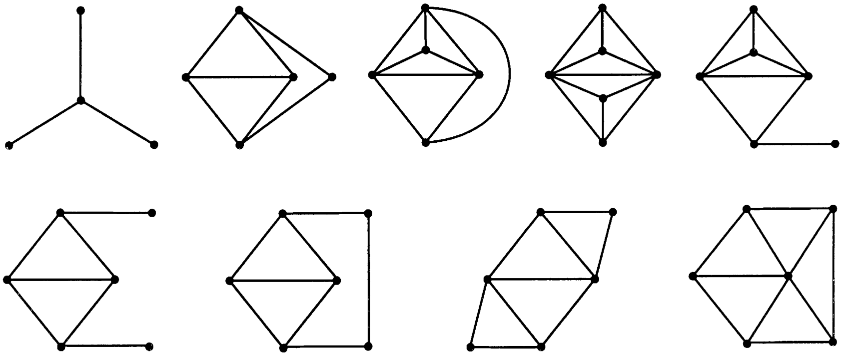

FIG. 2. The nine forbidden induced subgraphs.

implies that they are not in X. In addition, by applying Lemma 3.2 to the edge $a_x b_x$ we deduce that the degree of c_x is also greater than two and thus c_x is not in X either. Therefore, deleting X from G will not destroy any T_x. Now, for any triangle T of $G \backslash X$, if it contains an edge in $F = \{a_x b_x : x \in X\}$, then, by applying Lemma 3.6 to G and T we deduce that T does not contain any other edge in F. In conclusion, $G_X = G \backslash X$ is a rooted graph with the set of root edges F. Clearly, G is obtained from G_X by adding pendent triangles.

Notice that adding pendent triangles does not eliminate cut vertices and does not make disconnected graphs connected, it follows that, as G is 2-connected, G_X must be 2-connected as well.

It remains to prove that G_X is a line graph. Suppose it is not. We first observe from Beineke's Theorem and Lemma 3.8 that G_X has at least five vertices. Then we claim that, in G_X, every vertex in a diamond must have degree greater than two. Suppose u has degree two and is in a diamond (one of its triangles is uvw). Since u is not included in X, it must have degree three or more in G and thus must have a neighbor x in X. It follows that T_x is the triangle uvw. Let $G' = G \backslash (X - \{x\})$, the graph obtained from G_X by putting x back. Then we see that the edge xu is a component of $G' \backslash \{v, w\}$. Therefore, from Lemmas 3.2 and 3.7 we conclude that either (i) or (ii) holds for G, a contradiction and thus the claim is proved. It follows from this claim, Lemmas 3.3 and 3.7 that G_X has no diamonds.

Since G_X is not a line graph and it has no diamonds, by Beineke's Theorem, G_X has a claw. Since G_X has no Θ-graphs and W-graphs either, we deduce from Lemma 3.8 that G_X has a Θ_i-graph for $i = 1$ or 2. Then, by Lemmas 3.4 and 3.7 we may assume that $i = 2$. Let T be the inscribed triangle of this Θ_2-graph. By Lemmas 3.5–3.7, we deduce that $G_X \backslash T$ has an isolated vertex which has degree two in G_X; this vertex together with T induce a diamond in G_X, a contradiction. ∎

4. A PROOF OF THEOREM 1.1

Let $\mathscr{L}$ be the class of graphs that do not have Θ-, W-, and R-graphs. The next is the main result of this section, which clearly includes both Theorem 1.1 and Corollary 1.1.

THEOREM 4.1. *The following are equivalent for a graph G:*

(i) *G is CM*;
(ii) *G is cycle ideal*;
(iii) *G is in $\mathscr{L}$*;
(iv) *G is ESP.*

We will prove implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Again, we proceed by proving a sequence of lemmas.

LEMMA 4.1. *If G is cycle ideal and $U \subseteq V(G)$, then $G \backslash U$ is also cycle ideal.*

Proof. For any nonnegative integral function w on $V - U$, let us define w^+ on V with $w^+(x) = 0$ for all $x \in U$ and $w^+(x) = w(x)$ for all $x \in V - U$. Then we have

$$\nu_w^*(G \backslash U) = \nu_{w^+}^*(G) = \tau_{w^+}(G) = \tau_w(G \backslash U). \quad \blacksquare$$

It is worth pointing out that Lemma 4.1 is a very special case of a general result [2, 10] which says that all "minors" of an "ideal hypergraph" are still "ideal hypergraphs".

LEMMA 4.2. *If G is a Θ-graph, a W-graph, or an R-graph, then G is not a cycle ideal graph.*

Proof. It is clear from the definition of cycle ideal graphs that we only need to find a nonnegative integral function w on $V(G)$ such that $\nu_w^*(G)$ is not an integer. When w is given, the way we compute $\nu_w^*(G)$ is to use linear programming duality theorem [11]. For those who are not familiar with this theorem, here is a very brief outline. Let M be the cycle-vertex incidence matrix of G. If x and y are nonnegative vectors such that $y^T M \leqslant w^T$ and $Mx \geqslant e$, where e is the all-one vector, then x and y form a pair of *feasible solutions*. If, in addition, $w^T x = y^T e$, then $w^T x$ is called the *common value* of these solutions and the duality theorem guarantees that $\nu_w^*(G)$ is this common value.

First, let G be a Θ-graph and let u_1, u_2 be the two vertices of degree three. Set $w(v) = 2$ if $v \in \{u_1, u_2\}$ and $w(v) = 1$ otherwise. Let us choose three internal vertices v_1, v_2, v_3 from the three u_1u_2-paths, respectively. Define $x_v = \frac{1}{2}$ if $v \in \{v_1, v_2, v_3\}$ and $x_v = 0$ otherwise. Also define $y_C = \frac{1}{2}$ for all the three cycles C of G. Then it is straightforward to verify that x and y form a pair of feasible solutions with common value 3/2. Therefore, $v_w^*(G) = 3/2$, which is not integral under our choice of integral w.

Next, let G be a W-graph, with hub u, and let r be the degree of u. Set $w(v) = 1$ for all $v \in V(G)$. Define $x_u = 1 - \frac{2}{r}$, $x_v = \frac{1}{r}$ if $v \neq u$ has degree three, and $x_v = 0$ otherwise. Also define $y_C = \frac{1}{r}$ for each cycle C obtained by subdividing a triangle using u, $y_C = 1 - \frac{2}{r}$ for the only cycle C in $G \backslash u$, and $y_C = 0$ for all other cycles C. Then, like in the previous case, one can check that x and y form a pair of feasible solutions with common value $v_w^*(G) = 2 - \frac{2}{r}$, which is not integral for all $r \geqslant 3$.

Finally, let G be an R-graph. Set $w(v) = 1$ for all $v \in V(G)$. By Lemma 4.1 and the above discussion, we may assume that G does not have any Θ-graphs. Therefore, G is obtained from an odd cycle by replacing each edge $e = uv$ with either a cycle C_e containing e or two triangles uab, vcd together with two vertex-disjoint paths P_{ac} and P_{bd} between $\{a, b\}$ and $\{c, d\}$. In the former case, define $x_u = x_v = 1/2$ and $y_{C_e} = 1/2$; in the latter case, define $x_u = x_a = x_c = x_v = 1/2$ and $y_C = 1/2$ for the following three cycles C: $uabu$, $vcdv$ and $abP_{bd}dcP_{ac}a$. For all the remaining vertices v and all the remaining cycles C, we define $x_v = 0$ and $y_C = 0$. Again, it is routine to verify that x and y form a pair of feasible solutions with common value $t/2$, where t is the number of vertices v with $x_v = 1/2$. Suppose, when constructing G from an odd cycle C of length r, the number of edges of C that are not replaced by a cycle is s. Then it is not difficult to see that $t = r + 2s$. Consequently, $v_w^*(G) = t/2$ is not integral, which completes the proof. ∎

Lemma 4.3. *If $G \in \mathscr{L}$ is a 0- or 1-sum of G_1 and G_2, then both G_1 and G_2 are in $\mathscr{L}$.*

Proof. This is clear from the definition of $\mathscr{L}$ since both G_1 and G_2 are induced subgraphs of G. ∎

Lemma 4.4. *If a 2-connected graph $G \in \mathscr{L}$ is a 2-sum of two smaller graphs G_1 and G_2, then both G_1 and G_2 are in $\mathscr{L}$.*

Proof. Suppose some G_i, say G_1, has an induced subgraph H which is a Θ-graph, a W-graph, or an R-graph. We need to show that G has a Θ-graph, a W-graph, or an R-graph. Let x, y be the common vertices of G_1 and G_2, and let z_i be the only vertex in $G_i \backslash V(G)$. If $z_1 \notin V(H)$, then we are done since H is an induced subgraph of G. If $z_1 \in V(H)$, then both x and y are in H since H has minimum degree of at least two. Note that Θ-graphs do not have triangles and triangles in W-graphs only contain vertices of degree greater than two. Thus H can only be an R-graph. It follows from the 2-connectivity of G that $G_2\backslash z_2$ is also 2-connected. Therefore, xy is contained in an induced cycle C of $G_2\backslash z_2$. Now it is clear that $V(H\backslash z_1) \cup V(C)$ induces an R-graph in G, as required. ∎

LEMMA 4.5. *Let $G \in \mathscr{L}$ be a* 3-*sum of G_1 and G_2 over a triangle $x_1x_2x_3$. For $i = 1, 2,$ and $1 \leqslant j < k \leqslant 3$, let G_{ijk} be obtained from G_i by adding a new vertex x_{ijk} and two new edges $x_{ijk}x_j$ and $x_{ijk}x_k$. Then all G_{ijk} are in $\mathscr{L}$.*

Proof. Suppose to the contrary that some G_{ijk}, say G_{112}, has an induced subgraph H which is a Θ-graph, a W-graph, or an R-graph. We aim to show that G has a Θ-graph, a W-graph, or an R-graph. Like in the proof of the last lemma, we may assume that x_{112} is in H and H is an R-graph which contains the entire triangle $x_{112}x_1x_2$. As $x_1x_2x_3$ is a stable triangle in G_2, by definition there is a path, other than x_1x_2, in G_2 from x_1 to x_2 which avoids x_3; let P be such a path with the minimum length. Then $V(P)$ induces a cycle. Observe that x_3 is adjacent to no vertex in $P - \{x_1, x_2\}$, for otherwise $V(P) \cup \{x_3\}$ induces a W-graph in G, a contradiction. Hence, no vertex in $P - \{x_1, x_2\}$ is adjacent to any vertex in $H - \{x_{112}, x_1, x_2\}$ for G is a 3-sum of G_1 and G_2 over $x_1x_2x_3$. It follows that the graph obtained from H by replacing the path $x_1x_{112}x_2$ with P is an R-graph of G, a contradiction. ∎

LEMMA 4.6. *Let G' be the graph obtained from a graph G by subdividing an edge yz with a vertex x. If yz is contained in no triangle of G, then G is ESP provided G' is ESP.*

Proof. We first make the natural correspondence between cycles in G' and G more precise. For each cycle C in G', let $\phi(C)$ be the cycle in G such that, if x is not in C then $\phi(C) = C$, and if x is in C then $\phi(C)$ is obtained from $C\backslash x$ by adding the new edge yz. Then ϕ is a 1–1 mapping. Let $\mathscr{C}$ be a set of cycles in G. Define $\mathscr{C}' = \{\phi^{-1}(C) : C \in \mathscr{C}\}$. Since G' is ESP, $\mathscr{C}'$ has an equitable subpartition $(\mathscr{C}'_1, \mathscr{C}'_2)$. Now, for $i = 1, 2$, let $\mathscr{C}_i = \{\phi(C) : C \in \mathscr{C}'_i\}$. Then it is straightforward to verify that $(\mathscr{C}_1, \mathscr{C}_2)$ is an equitable subpartition of $\mathscr{C}$. ∎

A set F of edges in a graph G is called an *edge cut* if there is a partition (X, Y) of $V(G)$ such that $X \neq \emptyset \neq Y$ and F consists of precisely all edges between X and Y. If an edge cut has only one edge, then the edge is called a *cut edge*. It is well known, and it is also easy to prove, that an edge is a cut edge if and only if it is not contained in any cycle. The next lemma is an analog of this fact that concerns with minimal edge cuts of size two. We remark that this lemma holds not only for simple graphs but also for graphs with parallel edges and loops.

LEMMA 4.7. *Let e_1, e_2 be two distinct edges in a graph G. Then the following are equivalent*:

(i) $\{e_1, e_2\}$ *is a minimal edge cut*;

(ii) *neither of e_1 and e_2 is a cut edge and any cycle that contains one must also contain the other*;

(iii) *e_1 is not a cut edge and every cycle containing e_1 must also contain e_2.*

Proof. We first prove implication (i) $\Rightarrow$ (ii). From the minimality of $\{e_1, e_2\}$ it is clear that neither edge is a cut edge. Let (X, Y) be a partition of $V(G)$ such that e_1 and e_2 are the only two edges between X and Y. Then e_i $(i = 1, 2)$ is the only edge of $G \backslash e_{3-i}$ that is between X and Y. It follows that e_i is a cut edge of $G \backslash e_{3-i}$, and thus e_i is not contained in any cycle of $G \backslash e_{3-i}$. Therefore, in G, every cycle that contains e_i must also contain e_{3-i}, so (ii) is proved.

Implication (ii) $\Rightarrow$ (iii) is obvious and thus we turn to (iii) $\Rightarrow$ (i). Since e_1 is not a cut edge, it is contained in a cycle C. Then we deduce from the assumptions in (iii) that $e_2 \in E(C)$ and e_1 is not contained in any cycle of $G \backslash e_2$. Consequently, e_1 is a cut edge of $G \backslash e_2$ and thus $V(G \backslash e_2) = V(G)$ can be partitioned into X and Y such that e_1 is the only edge of $G \backslash e_2$ that is between X and Y. But e_1 is not a cut edge of G, it follows that e_2 is also between X and Y and thus $\{e_1, e_2\}$ is an edge cut. Recall that $e_1, e_2 \in E(C)$, so neither e_1 nor e_2 is a cut edge. Therefore, $\{e_1, e_2\}$ is a minimal edge cut. ∎

Two distinct edges are called in *series* if they form a minimal edge cut. Let us also consider every edge as being in series with itself. Then it is easy to see from Lemma 4.7(ii) that being in series is an equivalence relation. We call each equivalence class a *series family*. A series family is *trivial* if it has only one edge.

LEMMA 4.8. *Let $\{e\}$ be a trivial series family of G. Then either $e = xy$ is a cut edge, or $G \backslash e$ has two edge-disjoint xy-paths.*

Proof. Suppose e is not a cut edge. By Lemma 4.7(iii), every edge cut that contains e must contain three or more edges. Then the lemma follows from Menger's theorem. ∎

The next is a lemma on nontrivial series families, which is a strengthening of Problems 6.27 in [9].

LEMMA 4.9. *Let $F = \{e_1, e_2, \ldots, e_t\}$ be a nontrivial series family of a connected graph G. Then $G \backslash F$ has precisely t components $G_1, G_2, \ldots, G_t$ such* *that, after renaming the indices, each* e_i *is between* $V(G_i)$ *and* $V(G_{i+i})$, *where* $G_{t+1} = G_1$. *In addition, if* x *and* y *are the two vertices (which could be identical) in* G_i *that are incident with* e_{i-1} *and* e_i, *respectively, where* $e_0 = e_t$, *then* G_i *has two edge-disjoint* xy*-paths.*

Proof. We prove the two conclusions separately. In proving the first part, we will allow a graph to have parallel edges and even loops. Since $t > 1$, e_1 is not a cut edge and so it is contained in a cycle C. We prove by induction on $k = |E(G)| - E(C)$. If $k = 0$, then $G = C$ and the result is clear. If $k > 1$, let $e \in E(G) - E(C)$ and let G' be obtained from G by contracting e. It is clear from the definition of being in series that F is a series family of G'. Then the result follows from our induction hypothesis.

To prove the second part, let us assume that the conclusion is false and we will find a contradiction. Since G_i is connected, it follows from Menger's theorem that G_i has a cut edge, say e that separates x from y. Consequently, e is contained in all xy-paths in G_i. Now from our result on the first part we deduce that e is contained in all cycles that contains e_1. Then, by Lemma 4.7(iii), e and e_1 are in series, contradicting the fact that $e \notin F$. ∎

A graph is *subcubic* if its maximum degree is at most three. If a vertex x has degree three, then the subgraph formed by the three edges incident with x is called a *triad* with *center* x. We shall follow convention to let $L(H)$ stand for the line graph of a graph H.

LEMMA 4.10. *Let* G *be obtained from a rooted* 2*-connected line graph* $L(H)$ *by adding pendent triangles. Suppose* H *is subcubic and none of its cycles has chords. Then* G *is ESP if it has no R-graphs.*

Proof. Since removing isolated vertices from H will not change the assumption or the conclusion of the lemma, we may assume that H has no isolated vertices. Let us make some further observations about H.

(1) *H is connected and its only cut edges are the pendent edges*: This follows from our assumption that $L(H)$ is 2-connected.

(2) *Every nonpendent edge of H is contained in a nontrivial series family of H*: Assume the contrary: there exists a nonpendent edge $e = xy$ for which $\{e\}$ is a series family. It follows from (1) that e is not a cut edge. Therefore, by Lemma 4.8, $H \backslash e$ has two edge-disjoint xy-paths. Note that these two paths are internally vertex-disjoint because H is subcubic. Thus e is a chord of the cycle formed by these two paths, contradicting the last assumption of the lemma.

(3) *If F is a nontrivial series family of H with $|F| = k$ odd, then F has two incident edges xy and xz such that they are the only two edges of H that are incident with x*: Let us label the edges in F and components of $G\backslash F$ as in Lemma 4.9. In addition, for each i, let $e_i = x_i y_i$ with $x_i \in V(G_i)$ and $y_i \in V(G_{i+1})$, where the subscript is taken modulo k. Let I_t, where $t = 1, 2$, be the set of indices i for which G_i has t vertices, and let I_3 be the remaining indices. For each $i \in I_2$, it is clear that the only edge f_i of G_i shares a common end with e_{i-1} and e_i. For each $i \in I_3$, it can be seen from (1) that $y_{i-1} \neq x_i$. In addition, since G_i is subcubic, we deduce from Lemma 4.9 that G_i has a cycle C_i containing y_{i-1} and x_i. Assume $I_1 = \emptyset$. Let R be the subgraph of H induced by the union of $\{e_i : 1 \leqslant i \leqslant k\}$, $\{f_i : i \in I_2\}$, and $E(C_i)$, for all $i \in I_3$. Since no cycle of H contains chords, each of the two sections of C_i, for any $i \in I_3$, between y_{i-1} and x_i has length at least two, and thus $L(R)$ is an R-graph, a contradiction. It follows that $I_1 \neq \emptyset$ and therefore (3) is proved.

(4) *We may assume that H has no triangles*: Let $K_4\backslash e$ be obtained from K_4 by deleting an edge, $K_{2,3}^+$ be obtained from $K_{2,3}$ by adding an edge between the two vertices of degree three, and F_4 be obtained from a path on four vertices by adding a new vertex of degree four. Suppose T is a triangle of H. Since no cycle of H has a chord, the only paths between any two vertices of T are the two in T. It follows from (1) and (3) that H is obtained from T by adding at most two pendent edges. Thus $L(H)$ can only be K_3, $K_4\backslash e$ or F_4. Since G contains no R-graphs, we conclude that either G is in $\mathscr{G} = \{K_3, K_4\backslash e, F_4, K_{2,3}^+\}$ or G can be constructed from graphs in $\mathscr{G}$ by 2-sums. Observe that all graphs in $\mathscr{G}$ are ESP, so (4) follows from Corollary 2.3.

(5) *We may assume that each nontrivial series family of H contains an even number of edges*: For each nontrivial series family F with $|F|$ odd, let e_1^F and e_2^F be two edges as described in (3) and let H' be obtained from H by subdividing each e_1^F exactly once. Clearly, all nontrivial series families of H' are even and $L(H')$ can be considered as obtained from $L(H)$ by subdividing each edge $e^F = e_1^F e_2^F$ exactly once. Since, by (4), H has no triangle, e^F is not contained in any triangle of $L(H)$, and so it is not contained in any triangle of G either. Thus G', the graph obtained from G by subdividing each e^F exactly once, can also be obtained from $L(H')$ by adding pendent triangles (in the same way as getting G from $L(H)$). Since subdividing edges does not introduce R-graphs, G' has no R-graphs. In addition, by Lemma 4.6, G is ESP if G' is. Therefore, we can replace G and H by G' and H', respectively, and so (5) is proved.

Now let $\mathscr{C}$ be an arbitrary set of cycles of G. We prove in the following that $\mathscr{C}$ admits an equitable subpartition. We prove by induction on $k = |\mathscr{C}|$. The result is obvious if $k = 1$, so we assume that $k > 1$. Without loss of generality, we also assume that all cycles in $\mathscr{C}$ are chordless. Then we observe that there are three types of cycles in $\mathscr{C}$. The first type are cycles of length four or more which correspond to cycles of H, the second type are triangles of $L(H)$ which correspond to triads of H, and the third type are pendent triangles which correspond to pairs of edges xy, xz of H for which x, called the *center* of the pair, has degree three in H. Let $\mathscr{D}_1$ be the set of cycles C of H for which $L(C) \in \mathscr{C}$, let $\mathscr{D}_2$ be the set of triads T of H for which $L(T) \in \mathscr{C}$, and let $\mathscr{D}_3$ be the set of pairs $P = \{e, f\}$ of edges of H for which there is a vertex p of G of degree two such that $P^* = \{p, e, f\}$ induces a pendent triangle of $\mathscr{C}$.

(6) *For each* $P = \{e_1, e_2\} \in \mathscr{D}_3$, *we may assume that the triad* $T = \{e_1, e_2, e_3\} \notin \mathscr{D}_2$: Suppose we have both $P \in \mathscr{D}_3$ and $T \in \mathscr{D}_2$. Then we apply the induction hypothesis to $\mathscr{C} - \{P^*, L(T)\}$, which implies the existence of an equitable subpartition $(\mathscr{C}_1, \mathscr{C}_2)$ of $\mathscr{C} - \{P^*, L(T)\}$. Without loss of generality, let us assume that $d_{\mathscr{C}_1}(e_3) \leqslant d_{\mathscr{C}_2}(e_3)$. Then it is easy to verify that $(\mathscr{C}_1 \cup \{L(T)\}, \mathscr{C}_2 \cup \{P^*\})$ is an equitable subpartition of $\mathscr{C}$.

(7) *We may assume that cycles in* $\mathscr{D}_1$ *are pairwise vertex-disjoint*: Suppose some C_1 and C_2 in $\mathscr{D}_1$ have a vertex in common. Then they must have an edge in common, as H is subcubic. Since $C_1 \neq C_2$, we can find a maximal common path R of these two cycles. Let x_1, x_2 be the ends of R and let T_i be the triad with center x_i $(i = 1, 2)$. Let $\mathscr{C}' = (\mathscr{C} - \{L(C_1), L(C_2)\}) \cup \{L(T_1), L(T_2)\}$. Then, using the fact that each C_i is chordless, it is straightforward to verify that the four edges in $E(T_1) \cup E(T_2) - E(R)$ are all distinct. Therefore, $d_{\mathscr{C}'}(x) \leqslant d_{\mathscr{C}}(x)$ for all $x \in V(G)$. If each $L(T_i)$ appears precisely once in $\mathscr{C}'$, then we can replace $\mathscr{C}$ by $\mathscr{C}'$ and (7) follows since both $L(C_1)$ and $L(C_2)$ are cycles of length at least 4 in G; else, let $\bar{\mathscr{C}}$ be obtained from $\mathscr{C}'$ by removing each $L(T_i)$ with multiplicity 2, for $i = 1, 2$. Then the induction hypothesis guarantees the existence of an equitable subpartition of $\bar{\mathscr{C}}$; by introducing the corresponding $L(T_i)$ precisely once to each part of this subpartition, we get an equitable subpartition of $\mathscr{C}$. So we are done.

Let us contract each C in $\mathscr{D}_1$ into a vertex. Since, by Lemma 4.7(ii), every cycle is the disjoint union of series families, we conclude from (2), (5), and (7), that the resulting graph H' is bipartite. Let X_1, X_2 be the two color classes of H'. Then $\mathscr{D}_1$ is naturally partitioned into $\mathscr{D}_1^1$ and $\mathscr{D}_1^2$ such that each $\mathscr{D}_1^i$ contains those cycles in $\mathscr{D}_1$ that are contracted to a vertex in X_i. The partition (X_1, X_2) also induces a partition (V_1, V_2) of $V(H)$ such that each V_i contains vertices x for which either $x \in X_i$ or x is in some $C \in \mathscr{D}_1^i$. Then we partition $\mathscr{C}$ into $\mathscr{C}_1$ and $\mathscr{C}_2$ as follows. For each $C \in \mathscr{D}_1$, we put $L(C)$ in $\mathscr{C}_i$ if C is not in $\mathscr{D}_1^i$. For each $T \in \mathscr{D}_2$, we put $L(T)$ in $\mathscr{C}_i$ if the center of T is in V_i. For each $P \in \mathscr{D}_3$, we put P^* in $\mathscr{C}_i$ if the center of P is in V_i.

We prove that $(\mathscr{C}_1, \mathscr{C}_2)$ is an equitable subpartition of $\mathscr{C}$. Since $(\mathscr{C}_1, \mathscr{C}_2)$ is a partition of $\mathscr{C}$, it is clear that we only need to verify $\max\{d_{\mathscr{C}_1}(x), d_{\mathscr{C}_2}(x)\} \leqslant \lceil d_{\mathscr{C}}(x)/2 \rceil$ for all $x \in V(G)$. It follows from (6) and (7) that $d_{\mathscr{C}}(x) \leqslant 3$ for all $x \in V(G)$. Thus we only need to show that, if $d_{\mathscr{C}}(x) \geqslant 2$ then $d_{\mathscr{C}_i}(x) > 0$ for $i = 1, 2$. Observe that if a vertex of G is contained in two or more cycles of $\mathscr{C}$, this vertex must be an edge e of H. Also observe that there are only two kinds of edges in H: those between V_1 and V_2, and those with both ends in some V_i which are precisely those in some $C \in \mathscr{D}_1$. Then the result follows from (6) and the definition of $\mathscr{C}_1$ and $\mathscr{C}_2$. ∎

Proof of Theorem 4.1. Implication (iv) $\Rightarrow$ (i) is given by Corollary 2.1. Implication (i) $\Rightarrow$ (ii) is obvious because of inequality (1.3). Implication (ii) $\Rightarrow$ (iii) follows from Lemmas 4.1 and 4.2. It remains to prove implication (iii) $\Rightarrow$ (iv); we apply induction on $|V(G)|$. The case $|V(G)| = 1$ is trivial, so we proceed to the induction step. By Lemmas 4.3–4.5 and Corollaries 2.2–2.4, we may assume that G cannot be represented as a k-sum ($k = 0, 1, 2, 3$) of two smaller graphs (otherwise we are done by induction). Then, we conclude from Theorem 3.1 that G is obtained from a rooted 2-connected line graph $L(H)$ by adding pendent triangles. Since G contains no K_4, H is subcubic. Also note that H contains no cycle with chords, for otherwise a cycle together with a chord in H would correspond to a W-graph (which is a subdivision of a wheel with four spokes) in G, a contradiction. Now we deduce from Lemma 4.10 that G is ESP. ∎

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