Also available on SCIENCE DIRECT www.sciencedirect.com

European Journal of Combinatorics 24 (2003) 849–853

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Factorizations of an n-cycle into two n-cycles

Leonardo Cangelmi

Dipartimento di Scienze, Facoltà di Economia, Università "G. d'Annunzio", V.le Pindaro 42, 65127 Pescara, Italy

Received 26 February 2003; accepted 1 July 2003

Abstract

We prove by elementary combinatorial methods that the number of factorizations of an n-cycle (with n odd) into the product of two n-cycles is $2(n-1)!/(n+1)$. Moreover, we generalize our method to the factorization of an even permutation in S_n into the product of two n-cycles, and we present an algorithm giving all the factorizations of any odd permutation in S_{n+1} into the product of an $(n+1)$-cycle and an n-cycle, where the fixed element of the second permutation is given.

1. Introduction

If σ is an even permutation in S_n, it is interesting to study its factorizations into the product of two n-cycles. An argument attributed to Gleason (see [1]) proves that the number of such factorizations is positive. Gleason's argument also gives a recursive method to construct all of them (see [2]), but it is not suitable to obtain closed formulae for their number, not even in special cases. Nevertheless, several authors obtained such formulae in general or in special cases using character theory or combinatorial ideas.

In particular, it is known that the number of factorizations of an even n-cycle into the product of two n-cycles is $2(n-1)!/(n+1)$ (see [3–5, 7]). In this paper, we give a further proof of this formula, using elementary combinatorial arguments. We reduce the problem to the computation of the number of factorizations of an $(n+1)$-cycle into the product of an $(n+1)$-cycle and an n-cycle, where the fixed element of the second factor is given. We then rely on the fact that the number of factorizations of an odd permutation in S_n into the product of an n-cycle and an $(n-1)$-cycle equals $2(n-2)!$. For the latter fact, there exists a constructive proof due to Bertram and Wei [4], and to Machì [6].

Then, we describe the generalization of our method to the factorization of any even permutation in S_n into the product of two n-cycles. This enables us to compute the number

E-mail address: cangelmi@sci.unich.it (L. Cangelmi).

0195-6698/$ - see front matter
doi:10.1016/S0195-6698(03)00107-0

of such factorizations for some particular even permutations, different from n-cycles. Finally, we give a recursive procedure to find all the factorizations of type $(n+1)\cdot n$ of an odd permutation in S_{n+1}, where the fixed element of the second factor is given.

We adopt the following notation and convention. Permutations act on the left, so that the product $\alpha\beta$ considered as a function means $\beta \circ \alpha$, and the image of an element x under α is denoted by x^α. If n is a positive integer, we let $[n]$ and $[n]'$ denote respectively the sets $\{1,\ldots,n\}$ and $\{0,1,\ldots,n\}$, and S'_n denote the symmetric group on $[n]'$. We will generally use primes to denote conjugacy classes of S'_n.

2. Factorizations of type $n \cdot n$

Let $\sigma \in S_n$ be an even permutation. Define

$$F(\sigma) = \{(\alpha,\beta) \in C_n \times C_n : \sigma = \alpha\beta\},$$

where C_n is the conjugacy class of n-cycles in S_n. For brevity, we say that the elements of $F(\sigma)$ are the factorizations of type $n\cdot n$ of σ. Our first aim is to compute $|F(\sigma)|$ by means of elementary combinatorial methods, at least in the case σ is an n-cycle.

The main idea is the following. Set $\sigma' = (0\ n)\sigma$, where 0 is to be considered as an extra symbol. Note that σ' is obtained from σ by inserting 0 after n in the cycle of σ containing n and leaving all the other cycles of σ unchanged. Then σ' is odd and belongs to S'_n.

For $\tau \in S'_n$, let

$$F'(\tau) = \{(\gamma,\delta) \in C'_{n+1} \times C'_n : \tau = \gamma\delta,\ 0^\delta = 0\},$$

where C'_{n+1} and C'_n are respectively the conjugacy classes in S'_n of $(n+1)$-cycles and of permutations of type $n\cdot 1$. The elements of $F'(\tau)$ are the factorizations of τ into the product of an $(n+1)$-cycle on $[n]'$ and a permutation which is the product of an n-cycle on $[n]$ and the 1-cycle (0).

The first and fundamental result is the following.

Lemma 1. *For $\sigma \in S_n$, there is a bijection between $F(\sigma)$ and $F'(\sigma')$, given by $(\alpha,\beta) \mapsto ((0\ n)\alpha, (0)\beta)$.*

Proof. If $(\alpha,\beta) \in F(\sigma)$, then $\sigma' = (0\ n)\sigma = (0\ n)\alpha\beta = [(0\ n)\alpha][(0)\beta]$. Since $(0\ n)\alpha$ is an $(n+1)$-cycle and $(0)\beta$ fixes 0 and is of type $n\cdot 1$, the map is well-defined.
Conversely, let $(\gamma,\delta) \in F'(\sigma')$. Then, $\gamma(n) = 0$, since $\sigma'(n) = 0$ and δ leaves 0 fixed. Hence $(0\ n)\gamma = (0)\eta$ and $\delta = (0)\theta$, with η and θ being two n-cycles on $[n]$. Therefore, the inverse map is $(\gamma,\delta) \mapsto (\eta,\theta)$. $\square$

Hence, our problem is reduced to the computation of the number of factorizations of type $(n+1)\cdot n$ of an odd permutation σ', with 0 being the fixed element of the second factor. Moreover, note that if (γ,δ) is a factorization in $F'(\sigma')$, we obtain the corresponding factorization (η,θ) in $F(\sigma)$ by simply deleting 0 in both γ and δ.

For odd $\tau \in S_n$, let

$$G(\tau) = \{(\gamma,\delta) \in C_n \times C_{n-1} : \tau = \gamma\delta\}.$$

It is well known that $|G(\tau)| = 2(n-2)!$. This was proved by different methods, and among others there are also elementary and recursive proofs due to Bertram and Wei [4], and to Machì [6].

For our purpose, we need to consider the factorizations in $G(\tau)$ having the same fixed element in the second factor. So we define, for odd $\tau \in S_n$ and $i \in [n]$,

$$G_i(\tau) = \{(\gamma, \delta) \in G(\tau) : i^{\delta} = i\}.$$

Now, we have to distinguish between the case in which σ, and hence σ', is a cycle and the general case.

2.1. *Factorization of even n-cycles*

When σ is an even n-cycle, then n is odd, and σ' is an odd $(n+1)$-cycle. This case corresponds to considering odd n-cycles τ. In this case, we are able to compute the number of factorizations of type $n \cdot (n-1)$ with given fixed element i.

Proposition 2. *Let $\tau \in S_n$ be an odd n-cycle. Then:*

$$|G_i(\tau)| = \frac{2(n-2)!}{n}.$$

Proof. For $i, j \in [n]$, there is a bijection between $G_i(\tau)$ and $G_j(\tau)$. Indeed, some power τ^h of τ sends i to j. Then the map $(\gamma, \delta) \mapsto (\gamma^{\tau^h}, \delta^{\tau^h})$ gives the required bijection. Hence $G(\tau)$ is a dijoint union of n subsets with the same number of elements, and the claim follows. □

Putting together the previous results, we obtain the formula for the number of factorizations of an n-cycle into the product of two n-cycles.

Theorem 3. *Let $\sigma \in S_n$ be an even n-cycle. Then:*

$$|F(\sigma)| = \frac{2(n-1)!}{n+1}.$$

Proof. By Lemma 1, $|F(\sigma)| = |F'(\sigma')|$. Since σ' is an odd $(n+1)$-cycle on $n+1$ elements, by the definition of $F'(\sigma')$ and applying Proposition 2 to σ', we have $|F'(\sigma')| = 2(n-1)!/(n+1)$. □

2.2. *Factorization of even permutations*

In the general case, σ is an even permutation, so σ' is odd. The corresponding case we have to consider is that of a general odd permutation τ. We obtain the following result, generalizing Proposition 2.

Theorem 4. *Let τ be an odd permutation in S_n, and $i, j = 1, \ldots, n$.*

1. *If i and j belong to cycles of τ of the same length, then $|G_i(\tau)| = |G_j(\tau)|$.*
2. *Suppose that the partition type of τ is $1^{e_1} \cdots n^{e_n}$ (hence $\sum_{h=1}^{n} he_h = n$). For $h \in [n]$, let $G(\tau, h) = G_x(\tau)$, where x is any fixed element of $[n]$ belonging to a cycle of τ of length h (let $G(\tau, h) = \emptyset$ when $e_h = 0$). Then $|G(\tau)| = 2(n-2)! = \sum_{h=1}^{n} he_h |G(\tau, h)|$.*

Proof. If i and j belong to cycles of τ of the same length, then there exists $\theta \in S_n$ which exchanges the cycles containing i and j, leaves all the elements of the other cycles of τ fixed, and sends i to j. So $\tau^\theta = \tau$ and $i^\theta = j$, and the map $(\gamma, \delta) \mapsto (\gamma^\theta, \delta^\theta)$ is a bijection between $G_i(\tau)$ and $G_j(\tau)$.

The general formula follows by grouping together all the terms $|G_i(\tau)|$ with i in a cycle of length h. □

As a special case, we are able to compute the number of factorizations of type $n \cdot n$ of some even permutations different from n-cycles. First, we derive the corresponding result for the factorizations of type $n \cdot (n-1)$ with given fixed element of some odd permutations.

Corollary 5. *Let $\tau \in S_n$ be an odd permutation with partition type k^e (so that $ek = n$, and $e(k-1)$ is odd). Then $|G_i(\tau)| = \frac{2(n-2)!}{n}$, for all $i \in [n]$.*

Proof. It is a direct application of Theorem 4. The sum equals $ke|G_i(\tau)| = n|G_i(\tau)|$, for all $i \in [n]$. □

Now, we apply this result to σ', for σ of suitable type.

Corollary 6. *Let $\sigma \in S_n$ be an even permutation with partition type $(k-1) \cdot k^e$ (so that $k(e+1) - 1 = n$, and $e(k-1) + k$ is even). Then $|F(\sigma)| = 2(n-1)!/(n+1)$.*

Proof. By Lemma 1, $|F(\sigma)| = |F'(\sigma')|$. Up to conjugation by a transposition, we may assume that n belongs to the cycle of length $k-1$. Therefore, σ' is odd, and has all the cycles of the same length, k. Hence, by applying Corollary 5 to σ', we get $|F'(\sigma')| = 2(n-1)!/(n+1)$. □

3. Factorizations of type $(n + 1) \cdot n$

Since our problem of factoring an even permutation in S_n (and, in particular, an n-cycle) into the product of two n-cycles has been reduced to that of factoring an odd permutation in S'_n (in particular, an $(n+1)$-cycle) into the product of an $(n+1)$-cycle and an n-cycle with 0 as fixed element of the second factor, we present a recursive procedure to find all the factorizations of the latter kind.

This procedure is derived from the algorithm for the computation of *all* the factorizations of type $n \cdot (n-1)$ of a permutation in S_n, which can be found in [4] and [6], which, in turn, is based on Gleason's argument [1].

Proposition 7. *The following procedure recursively computes all the factorizations in $F'(\tau)$, for any odd $\tau \in S'_n$.*

Let $n \geq 1$, *and* $\tau \in S'_n$ *odd.*
If $0^\tau = 0$, *then there are no elements in* $F'(\tau)$.
If $n = 1$, *then* $\tau = (1\ 0)$ *and* $F'(\tau) = \{((1\ 0), (1))\}$.
If $n \geq 2$, *up to conjugation, we can assume that* $n^\tau = 0$.
For each $h \in [n-1]$, *with* $h \neq 0^\tau$, *let* $\tau(n\ h\ 0) = t(n)$:
then t *is an odd permutation on* $[n-1]'$ *which does not leave* 0 *fixed.*
For each factorization $(c, d) \in F'(t)$, *let* $\gamma = c(n\ 0)$ *and* $\delta = d(n\ h)$:
then $(\gamma, \delta) \in F'(\tau)$.

Proof. First, we show that the procedure is well defined. Assuming that $n \geq 2$ and that 0 is not a fixed element of τ, let $m \in [n]$ such that $m^\tau = 0$. Then $m \neq 0$ and we can replace τ with $\tau^{(m\ n)}$, so that $n^\tau = 0$. Now, for $h \in [n-1]$ with $h \neq 0^\tau$, the permutation $\tau(n\ h\ 0)$ is odd and leaves n fixed, so it can be written as $t(n)$, where t is an odd permutation on $[n-1]'$ and does not leave 0 fixed. Then, for any factorization $t = cd$, where $(c, d) \in F'(t)$, we can write

$$\tau = t(n\ 0\ h) = cd(n\ 0)(n\ h) = [c(n\ 0)][d(n\ h)],$$

since d and $(n\ 0)$ commute. Finally, $c(n\ 0)$ is an $(n+1)$-cycle and $d(n\ h)$ is the product of an n-cycle on $[n]$ and the 1-cycle (0).

On the other hand, we have to verify that the procedure gives all the factorizations in $F'(\tau)$. Assume $n \geq 2$ and $n^\tau = 0$, and let $(\gamma, \delta) \in F'(\tau)$. Put $h = n^\delta$, and note that $h \neq 0, n, 0^\tau$. Then, define c and d by the relations $c(n) = \gamma(n\ 0)$ and $d(n) = \delta(n\ h)$. It turns out that c is an n-cycle on $[n-1]'$ and d is the product of an $(n-1)$-cycle on $[n-1]$ and (0). Moreover, we have

$$c(n)d(n) = \gamma(n\ 0)d(n) = \gamma d(n)(n\ 0) = \gamma\delta(n\ h)(n\ 0) = \tau(n\ h\ 0).$$

Therefore, letting $\tau(n\ h\ 0) = t(n)$, we have $(c, d) \in F'(t)$ and clearly the given factorization (γ, δ) is obtained by the procedure for $h = n^\delta$ and (c, d) defined above.

Finally, we prove that for different h's or different factorizations of t we always obtain different factorizations of τ. Let (γ_1, δ_1) and (γ_2, δ_2) be two factorizations in $F'(\tau)$ obtained, respectively, by h_1 and (c_1, d_1), and by h_2 and (c_2, d_2). Suppose that $(\gamma_1, \delta_1) = (\gamma_2, \delta_2)$, and moreover that $n^\tau = 0$. It follows at once that $c_1 = c_2$, since $c_1(n) = \gamma_1(n\ 0) = \gamma_2(n\ 0) = c_2(n)$, and that $h_1 = h_2$, since $h_1 = n^{\delta_1} = n^{\delta_2} = h_2$. Then we get $d_1 = d_2$, by observing that $d_1(n) = \delta_1(n\ h_1) = \delta_2(n\ h_2) = d_2(n)$. $\square$

Remark. We observe that this procedure does not appear to be suitable to derive a closed formula for the number of factorizations in $F'(\tau)$, not even in special cases. The reason is that the possibilities for the choice of h in the steps of the recursion are $n-2$ or $n-1$ depending on whether 0^τ is different from n or not. Namely, in the first case the cycle of τ containing 0 has length greater than 2, while in the second case such a cycle is just $(n\ 0)$. In the course of the procedure, we always meet both cases, regardless of the type of the permutation τ we start with, and it is not possible to control this phenomenon at each step.

References

[1] D. Husemoller, Ramified coverings of Riemann surfaces, Duke Math. J. 29 (1962) 167–174.

[2] D. Walkup, How many ways can a permutation be factored into two n-cycles?, Discrete Math. 28 (1979) 315–319.

[3] G. Boccara, Nombre de representations d'une permutation comme produit de deux cycles de longuers donnes, Discrete Math. 29 (1980) 105–134.

[4] E. Bertram, V. Wei, Decomposing a permutation into two large cycles: an enumeration, SIAM J. Alg. Discrete Methods 1 (1980) 450–461.

[5] R. Stanley, Factorization of permutations into n-cycles, Discrete Math. 37 (1981) 255–262.

[6] A. Machì, Sur les representations des permutations impaires, European J. Combin. 13 (1992) 273–277.

[7] D. Zagier, On the distribution of the number of cycles of elements in symmetric groups, Nieuw Arch. Wisk. 13 (1995) 489–495.