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The independence fractal of a graph

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Abstract

Independence polynomials of graphs enjoy the property of essentially being closed under graph composition (or 'lexicographic product'). We ask here: for higher products of a graph G with itself, where are the roots of their independence polynomials approaching? We prove that in fact they converge (in the Hausdorff topology) to the *Julia set* of the independence polynomial of G, thereby associating with G a fractal. The question arises as to when these fractals are connected, and for graphs with independence number 2 we exploit the Mandelbröt set to answer the question completely.

Keywords: Graph; Independence; Polynomial; Roots; Fractal; Julia set; Mandelbröt set

1. Introduction

For a graph G and non-negative integer k, let i_k be the number of independent sets of vertices in G of cardinality k. The *independence polynomial* of G is the generating polynomial $i_G(x) = \sum_{k=0}^{\beta} i_k x^k$ for the sequence $\{i_k\}$, where β is the largest k for which $i_k > 0$ (the *independence number* of G).

Independence polynomials are particularly well behaved with respect to lexicographic product. For two graphs G and H, let $G[H]$ be the graph with vertex set $V(G) \times V(H)$ and such that vertex (a, x) is adjacent to vertex (b, y) if and only if a is adjacent to b (in G) or $a = b$ and x is adjacent to y (in H). The graph $G[H]$ is the *lexicographic product* (or *composition*) of G and H, and can be thought of as the

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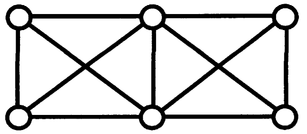

Fig. 1. The graph $P_3[P_2]$.

graph arising from G and H by substituting a copy of H for every vertex of G. The graph $P_3[P_2]$, for example, is shown in Fig. 1. A more general version of Theorem 1.1 was proved in [10], but for completeness we provide a short direct proof.

Theorem 1.1. *The independence polynomial of $G[H]$ is given by*

$$i_{G[H]}(x) = i_G(i_H(x) - 1). \tag{1}$$

Proof. By definition, the polynomial $i(G, i(H, x) - 1)$ is given by

$$\sum_{k=0}^{\beta_G} i_k^G \left(\sum_{j=1}^{\beta_H} i_j^H x^j \right)^k, \tag{2}$$

where i_k^G is the number of independent sets of cardinality k in G (similarly for i_k^H).

Now, an independent set in $G[H]$ of cardinality l arises by choosing an independent set in G of cardinality k, for some $k \in \{0, 1, \ldots, l\}$, and then, within each associated copy of H in $G[H]$, choosing a non-empty independent set in H, in such a way that the total number of vertices chosen is l. But the number of ways of actually doing this is exactly the coefficient of x^l in (2), which completes the proof. $\square$

As is the case with chromatic polynomials (cf. [7,16]), matching polynomials [11,12] and others, it is natural to consider the nature and location of the roots. Interesting in their own right, they can shed some light on the underlying combinatorics as well. It was conjectured in [6], for instance, that the independence vector $(i_0, i_1, \ldots, i_\beta)$ of any *well-covered* graph is unimodal (i.e., first non-decreases, then non-increases), and some partial results in that regard have been obtained via roots of independence polynomials [6]. Further results on independence polynomials and their roots can be found in [6,10,13,14].

It is easily verified that lexicographic product is an associative operation, and so we may speak of powers $G^k = \underbrace{G[G[G\cdots]]}_{k}$ of a graph G without ambiguity ($G^1 = G$). For $G = P_3$, a path on three vertices, the independence roots of G^{11} are shown in Fig. 2. It appears that the independence roots of G^k are approaching a fractal-like object as $k \to \infty$. We ask:

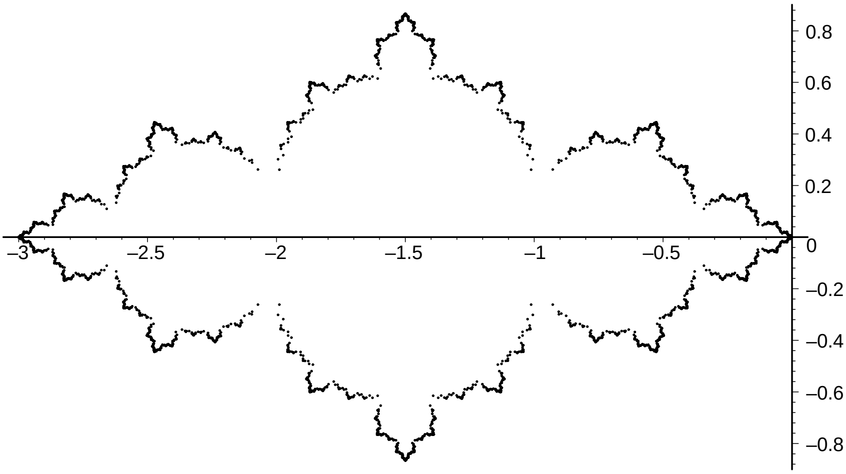

Fig. 2. The independence roots of G^{11}, where $G = P_3$.

Question 1.2. *For a graph G, what happens to the roots of the independence polynomials $i_{G^k}(x)$ as $k \to \infty$?*

A complete answer to Question 1.2 was provided by one of the authors in his Ph.D. thesis [15], where a fair amount of technical detail arose from the fact that independence polynomials are not quite closed under composition (cf. Eq. (1)). We can avoid this complication here by working with a slightly modified independence polynomial. Specifically, as there is but one independent set of cardinality 0 (the empty set), every independence polynomial has constant term 1. Define the *reduced* independence polynomial of G as the function $f_G(x) = i_G(x) - 1$, that is, $f_G(x) = \sum_{k=1}^{\beta} i_k x^k$. Eq. (1) then has the simple form

$$f_{G[H]}(x) = f_G(f_H(x)).$$

In this paper, we will answer Question 1.2 for the reduced independence polynomial $f_G(x)$, and indicate what small amendments to the result provide the answer for $i_G(x)$. The organization of the paper is as follows. Section 2 contains relevant background material from iteration theory. Incidentally, while Theorem 2.3 will have most direct application for us, it cannot (as far as the authors are aware) be found explicitly in the literature. In Section 3, we prove the main result (Theorem 3.3), which describes precisely where the reduced independence roots of powers G^k are approaching, and in what sense they do so. The upshot is an association of a fractal with G. We are led to ask for when these fractals are connected, and prove a result (Theorem 3.8) which implies that there are many connected graphs with disconnected fractals. In Section 4, we exploit the *Mandelbröt set* to decide which graphs of independence number 2 have a connected fractal, and we employ a different technique in Section 5 to answer the same question for some families of graphs of arbitrarily high independence numbers.

2. Background: Julia sets and iteration of polynomials

The field of complex analytic dynamics entails a study of iterating rational functions over the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ endowed with the spherical metric. Since we will be working exclusively with polynomials in this paper, we can get by with $\mathbb{C}$ together with the absolute value metric $|\cdot|$, which measures the distance between two points z and w as $|z-w|$. Except where otherwise stated, any definition or assertion made in this section can be found in Beardon's book [1]; much of the information can also be found in the works of Blanchard [2] and Brolin [5].

For a polynomial f and positive integer k, denote by $f^{\circ k}$ the map $f \circ f \circ \cdots \circ f$. Set $f^{\circ(0)}$ as the identity map, and $f^{\circ(-k)}$ as $f^{\circ(-1)} \circ f^{\circ(-1)} \circ \cdots \circ f^{\circ(-1)}$, where $f^{\circ(-1)}$ is the *set-valued* inverse of f, i.e., for $A \subseteq \mathbb{C}$, $f^{\circ(-1)}(A) = \{z \in \mathbb{C} : f(z) \in A\}$. (The symbol $\circ$ in the exponent serves to distinguish a *composition* power from a multiplicative one.)

2.1. Forward orbits

For a point $z_0 \in \mathbb{C}$, its *forward orbit* with respect to f is the set

$$\mathcal{O}^+(z_0) = \{f^{\circ k}(z_0)\}_{k=0}^{\infty}.$$

Definition 2.1. For a polynomial f, its filled Julia set $K(f)$ is the set of all points z whose forward orbit $\mathcal{O}^+(z)$ is bounded in $(\mathbb{C}, |\cdot|)$. Its Julia set $J(f)$ is the boundary, $\partial K(f)$, and its Fatou set $F(f)$ is the complement of $J(f)$ in $\mathbb{C}$.

The Julia set of $f(x) = 3x^3 + 9x^2 + 7x$ is shown (in black) in Fig. 3. A method for generating pictures of Julia sets is suggested by Theorem 2.2.

For the remainder of this section, we shall assume that f is a polynomial of degree at least 2.

As it turns out, $F(f)$ is an open subset of $(\mathbb{C}, |\cdot|)$, while $J(f)$ is compact in $(\mathbb{C}, |\cdot|)$. The latter implies that Julia sets of polynomials are *bounded* in $(\mathbb{C}, |\cdot|)$. The set $J(f)$ is infinite; in fact, $J(f)$ is a *perfect* set in that it is equal to its set of accumulation points. The sets $K(f)$, $J(f)$ and $F(f)$ are each *completely invariant* under f, that is, if A is any one of those sets, then $f(A) = A = f^{\circ(-1)}(A)$. Further, for any positive integer k, $F(f^{\circ k}) = F(f)$ and $J(f^{\circ k}) = J(f)$.

Periodic points play an important role in iteration theory. A point z_0 is a *periodic point* of f if, for some positive integer k, $f^{\circ k}(z_0) = z_0$. The smallest such k is the *period* of z_0 (and if $k=1$ then z_0 is, of course, a *fixed point* of f). The forward orbit of a periodic point z_0 is a *cycle*; if k is the period of the cycle, then the number

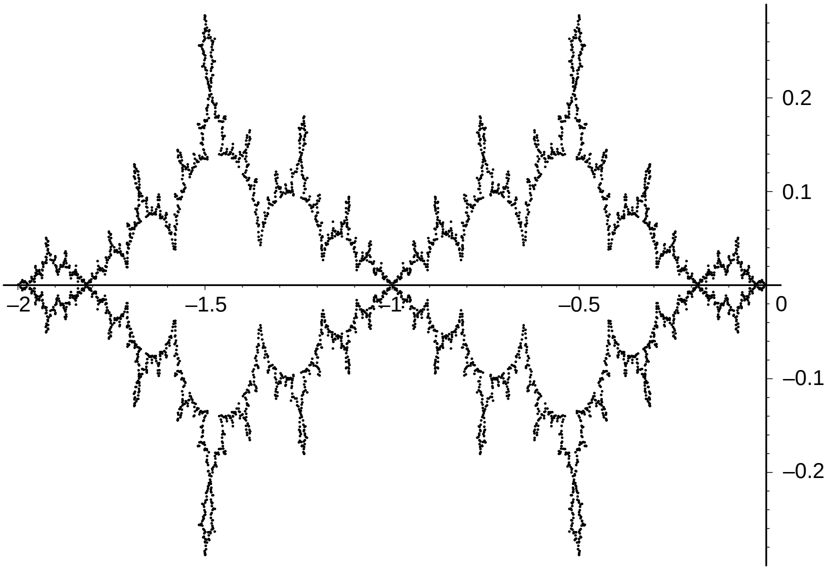

Fig. 3. Julia set of $3x^3 + 9x^2 + 7x$.

$\lambda = (f^{\circ k})'(z_0)$ is the *multiplier* of the cycle, and is independent of the choice of z_0 from the cycle. The cycle is

(i) *attracting* if $0 < |\lambda| < 1$,
(ii) *repelling* if $|\lambda| > 1$,
(iii) *rationally indifferent* if λ is a root of unity, and
(iv) *irrationally indifferent* if $|\lambda| = 1$ but λ is not a root of unity.

A basic (and non-trivial) fact is that

(i) attracting cycles lie in $F(f)$,
(ii) repelling cycles lie (and are dense) on $J(f)$,
(iii) rationally indifferent cycles lie on $J(f)$, and
(iv) an irrationally indifferent cycle may lie in either $F(f)$ or $J(f)$.

2.2. Backward orbits

For $z_0 \in \mathbb{C}$, its *backward orbit* with respect to f is the set

$$O^-(z_0) = \bigcup_{k=0}^{\infty} f^{\circ(-k)}(z_0).$$

A polynomial f has at most one *exceptional* point whose backward orbit is finite. For example, if $f(x) = x^n$ then 0 is exceptional as $O^-(0) = \{0\}$. In general an exceptional point, if it exists, lies in $F(f)$. The following fundamental result implies that the backward orbit of any *non*-exceptional point accumulates on $J(f)$. The symbol Cl denotes topological closure.

Theorem 2.2. (cf. Beardon [1]). *For a polynomial f of degree at least* 2,

(i) *if z_0 is non-exceptional then* $J(f) \subseteq \mathrm{Cl}(O^-(z_0))$;
(ii) *if $z_0 \in J(f)$ then* $J(f) = \mathrm{Cl}(O^-(z_0))$.

Intuitively, as $J(f)$ is a repelling set for f, it is somehow attracting for $f^{\circ(-1)}$. Instead of looking at the entire inverse orbit $O^{-1}(z_0)$, we could ask whether the sets $f^{\circ(-k)}(z_0)$ *converge* (in some sense) to $J(f)$. Indeed, they do: Hickman [15] established the following result, of which we will make important use in Section 3. The *Hausdorff metric* measures the distance between two compact subsets A and B of $(\mathbb{C}, |\cdot|)$ as $h(A,B) = \max(d(A,B), d(B,A))$, where $d(A,B) = \max_{a \in A} \min_{b \in B} |a-b|$. Since the sets $f^{\circ(-k)}(z_0)$ are finite, they are necessarily compact.

Theorem 2.3. (Hickman [15]). *Let f be a polynomial, and z_0 a point which does not lie in any attracting cycle or Siegel disk of f. Then*

$$\lim_{k \to \infty} f^{\circ(-k)}(z_0) = J(f),$$

where the limit is taken with respect to the Hausdorff metric on compact subsets of $(\mathbb{C}, |\cdot|)$.

We need not discuss Siegel disks here; it suffices to mention that they are contained in $F(f)$. As attracting cycles also lie in $F(f)$, it follows immediately from Theorem 2.3 that $\lim_{k \to \infty} f^{\circ(-k)}(z_0) = J(f)$ for any point $z_0 \in J(f)$. For the sake of completeness, the proof of Theorem 2.3 (extracted from [16]) is included in Appendix A.

2.3. Conjugacy

A *Möbius map* is a rational map of the form

$$\phi(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

where a, b, c and d are fixed complex numbers. The condition $ad - bc \neq 0$ ensures that ϕ is one to one and thus invertible. Two polynomials f and g are *conjugate* if there exists a Möbius map ϕ such that

$$g = \phi \circ f \circ \phi^{\circ(-1)}.$$

It follows easily that, for any positive integer k,

$$g^{\circ k} = \phi \circ f^{\circ k} \circ \phi^{\circ(-1)},$$

an important property of conjugacy. Julia sets of conjugate polynomials are related in the following manner.

Theorem 2.4. (cf. Beardon [1]). *If $g = \phi \circ f \circ \phi^{\circ(-1)}$ for some Möbius map ϕ, then $F(g) = \phi(F(f))$ and $J(g) = \phi(J(f))$. The sets $J(g)$ and $J(f)$ are then said to be analytically conjugate, as are $F(g)$ and $F(f)$.*

3. Independence fractals of graphs: a general theory

We set out now to describe just where the *reduced* independence roots of powers G^k (i.e., the roots of $f_{G^k} = f_G{}^{\circ k}$) of a graph G are approaching as $k \to \infty$. The upshot will be an association of a fractal with G. For each $k \geqslant 1$ the set, Roots (f_{G^k}), of roots of f_{G^k} is a finite—and therefore compact—subset of $(\mathbb{C}, |\cdot|)$. We ask then whether the *limit* of the sequence $\{\text{Roots } (f_{G^k})\}$ exists in general, with respect to the Hausdorff metric (cf. Section 2) on compact subsets of $(\mathbb{C}, |\cdot|)$. In fact, it does.

Definition 3.1. The independence fractal of a graph G is the set

$$\mathscr{F}(G) = \lim_{k \to \infty} \text{Roots } (f_{G^k}). \tag{3}$$

That $\mathscr{F}(G)$ actually exists for every graph G is part of Theorem 3.3, the main result of this section. We begin with a simple but important characterization of the right-hand side of Eq. (3). For each $k \geqslant 2$, associativity of graph composition allows us to write $G^k = G^{k-1}[G]$, and Proposition 1 then implies that

$$f_{G^k} = f_{G^{k-1}} \circ f_G,$$

which in turn leads to the relation

$$\text{Roots } (f_{G^k}) = f_G{}^{\circ(-1)}(\text{Roots } (f_{G^{k-1}})).$$

Also, note that

$$\text{Roots } (f_G) = f_G{}^{\circ(-1)}(0).$$

Hence,

Proposition 3.2. *For each $k \geqslant 1$, we have*

$$\text{Roots } (f_{G^k}) = f_G{}^{\circ(-k)}(0). \tag{4}$$

Therefore,

$$\mathscr{F}(G) = \lim_{k \to \infty} f_G{}^{\circ(-k)}(0). \tag{5}$$

An application of Theorem 2.3 will then complete the picture; Theorem 3.3 answers completely our question in general. For the graph $G = K_1$, $f_G(x) = x$ and $f_{G^k}(x) = x$ for all k, whence $\mathscr{F}(G) = \{0\}$.

Theorem 3.3. *The independence fractal $\mathscr{F}(G)$ of a graph $G \neq K_1$ is precisely the Julia set $J(f_G)$ of its reduced independence polynomial $f_G(x)$. Equivalently, $\mathscr{F}(G)$ is the closure of the union of the reduced independence roots of powers G^k, $k = 1, \ldots, \infty$.*

Proof. If G has independence number 1, then $G = K_n$ for some $n \geqslant 2$, and $f_G(x) = nx$, whose Julia set is $\{0\}$ (as any non-zero point has an unbounded forward orbit). Now, $G^k = K_{n^k}$ for all k, and $f_{G^k}(x) = n^k x$, whose set of roots is $\{0\}$. The union and limiting root set is therefore $\{0\} = J(f_G)$, and the result holds.

If G has independence number at least 2, then $f_G(x)$ has degree at least 2. Since $f_G(x) = \sum_{k=0}^{\beta} i_k x^k$, we have that $f_G(0) = 0$ and $f'_G(0) = i_1 = |V(G)| > 1$. Thus, 0 is a *repelling* fixed point of $f_G(x)$, and therefore lies in $J(f_G(x))$. In particular, $z_0 = 0$ satisfies the hypothesis of Theorem 2.3 (cf. the remarks immediately following that theorem), and so

$$\lim_{k \to \infty} {f_G}^{\circ(-k)}(0) = J(f_G). \tag{6}$$

The left-hand side is $\mathscr{F}(G)$, by Eq. (5). That $\mathscr{F}(G) = \mathrm{Cl}(\bigcup_{k \geqslant 1} \text{Roots } (f_{G^k}))$ then follows from Eq. (4) and Theorem 2.2 (ii), together with the facts that $0 \in J(f_G)$ and $\mathscr{F}(G) = J(f_G)$. $\square$

Some remarks are in order.

Remark 1. Since $f_G(0) = 0$, we have $0 \in {f_G}^{\circ(-1)}(0)$. Applying ${f_G}^{\circ(-1)}$ to both sides yields ${f_G}^{\circ(-1)}(0) \subseteq {f_G}^{\circ(-2)}(0)$, and, by induction, ${f_G}^{\circ(-k)}(0) \subseteq {f_G}^{\circ(-(k+1))}(0)$ for all k. Hence, for all k, Roots $(f_{G^{k+1}}) \supseteq$ Roots (f_{G^k}).

Remark 2. For the 'usual' independence polynomials $i_{G^k}(x) = f_{G^k}(x) + 1$, the limiting root set is $\mathscr{I}(G) = \lim_{k \to \infty} {f_G}^{\circ(-k)}(-1)$, which always *contains* $\mathscr{F}(G) = J(f_G)$. The containment is proper exactly when i_G has -1 as a root of multiplicity at least 2, since then -1 is an *attracting* fixed point of f_G. The situation there is that Roots $(i_{G^{k+1}}) \supseteq$ Roots (i_{G^k}) for all k, and $\mathscr{I}(G)$ is *partitioned* by the set, $\bigcup_{k \geqslant 1}$ Roots (i_{G^k}), and its set of accumulation points, $J(f_G)$. However, the 'new' independence roots at each step, namely Roots $(i_{G^{k+1}}) \backslash$Roots (i_{G^k}), converge to precisely $\mathscr{F}(G) = J(f_G)$. All of these assertions are proved in [15], where $\mathscr{I}(G)$ is called the independence *attractor* of G, while $\mathscr{F}(G)$ is denoted by $\tilde{\mathscr{I}}(G)$ and (as here) is the independence *fractal* of G.

Remark 3. Actually, the connection between $\mathscr{F}(G)$ and $\mathscr{I}(G)$ described in Remark 2 *fails* to hold precisely when G is *empty*, where there are no new independence roots at *any* step. Indeed, for $G = \overline{K_n}$ we have $i_G(x) = (1+x)^n$, and since for each k, $G^k = \overline{K_{n^k}}$, $i_{G^k}(x) = (1+x)^{n^k}$, whose only root is -1. Thus, $\mathscr{I}(G) = \{-1\}$. Here, $\mathscr{F}(G) \nsubseteq \mathscr{I}(G)$: If $n = 1$ then for all k, $G^k = K_1$ and $f_{G^k}(x) = x$, whence $\mathscr{F}(G) = \{0\}$; if $n > 1$ then the roots of $f_{G^k} = (1+x)^{n^k} - 1$ become dense on the circle $|z+1| = 1$ as $k \to \infty$, and by Theorem 3.3, $\mathscr{F}(G)$ is precisely that circle, which is also $J((1+x)^n - 1)$.

Since empty graphs have been analyzed completely in Remark 3, and, moreover, are the only source of discrepancy for the connection between $\mathscr{F}(G)$ and $\mathscr{I}(G)$ described in Remark 2:

We will assume henceforth that G is a non-empty graph.

Now, as Julia sets are typically fractals, we are in essence associating a fractal $\mathscr{F}(G)$ with a graph G. The question arises as to the possible connections between the two objects. How are graph–theoretic properties encoded in the fractals? What does $\mathscr{F}(G)$ say about G itself? In particular, in the sections which follow we will come across independence fractals that are connected, and others that are disconnected. We ask here:

Question 3.4.. *For which graphs G is $\mathscr{F}(G)$ connected?*

Remark 4. Even for the usual independence polynomials, Question 3.4 is the right one to ask: When -1 has multiplicity at most 1 as a root of i_G, then $\mathscr{I}(G)$ and $\mathscr{F}(G)$ are equal anyway [15]. Moreover, when i_G has -1 as a root of multiplicity at least 2, then the nature of the resulting partition of $\mathscr{I}(G)$ (described in Remark 2) immediately implies that $\mathscr{I}(G)$ is disconnected. What is more interesting is whether its set of accumulation points (equivalently, the limiting set for the *new* independence roots at each step), $\mathscr{F}(G) = J(f_G)$, is connected.

We will prove momentarily that, in fact, *every* graph—with the exception of complete graphs—is contained, as an induced subgraph, in a graph with the same independence number, having a disconnected independence fractal. The following result from iteration theory, which links the critical points of a polynomial to the connectivity of its Julia set, will be useful. A *totally disconnected* set is one whose components (maximal connected subsets) contain just one point.

Theorem 3.5. (cf. Beardon [1]).*Let f be a polynomial of degree at least two.*

- *Its Julia set $J(f)$ is connected if and only if the forward orbit of each of its critical points is bounded in* $(\mathbb{C}, |\cdot|)$.
- *Its Julia set $J(f)$ is totally disconnected if* (*but not only if*) *the forward orbit of each of its critical points is unbounded in* $(\mathbb{C}, |\cdot|)$.

With Theorem 3.5 at hand, we prove:

Theorem 3.6. *Every graph G with independence number at least two is an induced subgraph of a graph H with the same independence number*, *whose independence fractal is disconnected.*

Proof. Since $f_G(x)$ has degree at least 2, a simple argument using the triangle inequality shows that there exists a real number $R>1$ such that $|z|>R \Rightarrow |f_G(z)|>2|z|$, which implies that the forward orbit of z is unbounded in $(\mathbb{C}, |\cdot|)$.

Now, not every critical point of f_G is a root of f_G. Indeed, for a root r of both f'_G and f_G, its multiplicity as a root of f_G is one greater than its multiplicity as a root of f'_G. But $\deg f_G = \deg f'_G + 1$, and so, if every critical point of f_G were a root of f_G, then in fact f_G must have only one critical point c, and $f_G(x) = a(x+c)^\beta$. But we know that $x | f_G(x)$, and so $c = 0$ and $f_G(x) = ax^\beta$. This could only be the case if $\beta = 1$, which it is not.

Let c then be a critical point of f_G for which $f_G(c) = w \neq 0$, and choose a positive integer p large enough that $|p \cdot w| > R$. For the graph $G[K_p]$, we have $f_{G[K_p]}(x) = f_G(px)$, a critical point of which is c/p. But then $f_{G[K_p]}(c/p) = f_G(c) = w$, and $|{f_{G[K_p]}}^{\circ k}(w)| = |{f_G}^{\circ k}(pw)| \to \infty$ as $k \to \infty$. Hence, by Theorem 3.5, the graph $G[K_p]$, which has independence number β, and of which G is an induced subgraph, has a disconnected independence fractal. $\square$

We proved that $G[K_p]$ has a disconnected independence fractal for all sufficiently large p. In fact, the same is also true of $K_p[G]$, for since $f_{K_p}(x) = px$, we have:

Theorem 3.7. *For a graph G and positive integer p,*

$$f_{K_p[G]}(px) = p \cdot f_G(px) = p \cdot f_{G[K_p]}(x).$$

That is,

$$f_{K_p[G]} \circ \phi = \phi \circ f_{G[K_p]},$$

where ϕ is the Möbius map $x \mapsto px$. Hence,

$$\mathscr{F}(K_p[G]) = p \cdot \mathscr{F}(G[K_p]).$$

The last line follows directly from Theorem 2.4 on Julia sets of conjugate polynomials, and tells us that the independence fractal of $K_p[G]$ is a mere scaling of that of $G[K_p]$. Thus, the former set must also be disconnected for all sufficiently large p, and so we have the following result, which at the very least suggests that graph connectedness and connectedness of independence fractals are not related.

Theorem 3.8. *If G is a graph with independence number at least* 2, *then for all sufficiently large p, the join of p copies of G has a disconnected independence fractal.*

Graphs with independence number 1 are not very interesting, since $f_{K_n}(x) = nx$, whose Julia set is just $\{0\}$. For graphs with independence number 2, we can exploit the Mandelbröt set to decide when their independence fractals are connected; this is the subject of Section 4. In Section 5, we will analyze two families of graphs with arbitrarily high independence numbers.

4. Graphs with independence number 2

For a graph G with independence number 2, having n vertices and m non-edges (i.e., $\bar{G}$ has exactly m edges), its independence polynomial is

$$f_G(x) = mx^2 + nx. \tag{7}$$

The *Mandelbröt set* $\mathscr{M}$ is the set of all complex numbers c for which the Julia set of the polynomial $x^2 + c$ is connected. For any other value of c, $J(x^2 + c)$ is not only disconnected, but *totally disconnected*, as $x^2 + c$ has only one critical point (cf. Theorem 3.5). Julia sets of this type are often called *fractal dust*. A plot of the Mandelbröt set (a subset of the complex c-plane) is shown in Fig. 4. A well-known fact (cf. [10]) is that $\mathscr{M}$ is contained in the disk $|c| \leqslant 2$.

Let us then consider a polynomial of the form $x^2 + c$ to which $f_G(x)$ is conjugate. It is straightforward to check that

$$g_G = \phi \circ f_G \circ \phi^{\circ(-1)},$$

where

$$g_G(x) = x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2 \tag{8}$$

and

$$\phi(x) = mx + \frac{n}{2}. \tag{9}$$

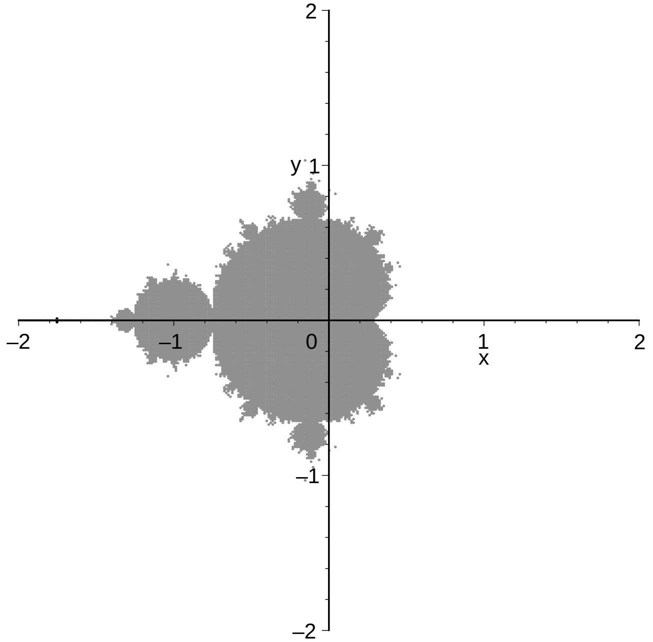

Fig. 4. The Mandelbröt set.

Then

$$\phi^{\circ(-1)}(x) = \frac{1}{m}x - \frac{n}{2m} \tag{10}$$

and (from Theorem 2.4)

$$\mathscr{F}(G) = \phi^{\circ(-1)}(J(x^2 + c)),$$

where $c = \frac{n}{2} - (\frac{n}{2})^2$. Thus, $\mathscr{F}(G)$ is a mere scaling and shifting of $J(x^2 + c)$, and since c is independent of m, the connectivity of $\mathscr{F}(G)$ depends only on how many *vertices* G has; the fact that G is non-empty implies that $n \geqslant 3$. The location of $\mathscr{F}(G)$, though, depends on both the numbers of vertices and edges in G, as Theorems 4.2–4.4 imply the following:

Theorem 4.1. *If G is a non-empty graph with independence number* 2 *having n vertices and m non-edges, and* $z \in \mathscr{F}(G)$, *then*

(i) $-\frac{n}{m} \leqslant \mathscr{Re}(z) \leqslant 0$, *and*
(ii) $\mathscr{Im}(z) = 0$, *unless* $n = 3$, *in which case* $-\dfrac{\sqrt{3}}{2m} \leqslant \mathscr{Im}(z) \leqslant \dfrac{\sqrt{3}}{2m}$.

4.1. *Graphs for which* $\beta = 2$, $n = 3$

There are exactly two graphs with independence number 2 on $n = 3$ vertices, namely $K_1 \uplus K_2$, the disjoint union of a point and an edge, and P_3, the path on three vertices. Their reduced independence polynomials are $f_{K_1 \uplus K_2}(x) = 2x^2 + 3x$ and $f_{P_3}(x) = x^2 + 3x$; thus, $\mathscr{F}(K_1 \uplus K_2) = J(2x^2 + 3x)$ and $\mathscr{F}(P_3) = J(x^2 + 3x)$.

For either graph G, Eq. (8) says that $f_G(x)$ is conjugate to the polynomial $g_G(x) = x^2 - \frac{3}{4}$. For $G = K_1 \uplus K_2$, Eq. (10) tells us that $\phi^{\circ(-1)}(x) = \frac{1}{2}x - \frac{3}{4}$, while for $G = P_3$, $\phi^{\circ(-1)}(x) = x - \frac{3}{2}$. Since $-\frac{3}{4}$ lies in the Mandelbröt set, $J(x^2 - \frac{3}{4})$ is connected. By Theorem 2.4, $\mathscr{F}(G) = \phi^{\circ(-1)}J(x^2 - \frac{3}{4})$, and since, for either graph G, $\phi^{\circ(-1)}$ is a mere scaling and shifting, $\mathscr{F}(G)$ must also be connected.

With a little work, we can determine a box containing $J(x^2 - \frac{3}{4})$. We prove in appendix that $J(x^2 - \frac{3}{4})$ is contained in the box $[-\frac{3}{2}, \frac{3}{2}] \times [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$, and that the box is tight. Applying $\phi^{\circ(-1)}$ to this box then gives a tight box containing $\mathscr{F}(G)$. We have proved:

Theorem 4.2. *If G is a graph with independence number* 2 *on* $n = 3$ *vertices, then*

$$\mathscr{F}(G) = \phi^{\circ(-1)}\left(J\left(x^2 - \frac{3}{4}\right)\right),$$

where either

(i) $G = K_1 \uplus K_2$ *and* $\phi^{\circ(-1)}(x) = \frac{1}{2}x - \frac{3}{4}$, *or*
(ii) $G = P_3$ *and* $\phi^{\circ(-1)}(x) = x - \frac{3}{2}$.

Therefore, $\mathscr{F}(G)$ *is connected, and*

(i) $\mathscr{F}(G = K_1 \uplus K_2) = J(2x^2 + 3x) \subseteq [-\frac{3}{2}, 0] \times [-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}]$, *while*
(ii) $\mathscr{F}(G = P_3) = J(x^2 + 3x) \subseteq [-3, 0] \times [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$.

Plots of $\mathscr{F}(K_1 \uplus K_2)$ and $\mathscr{F}(P_3)$ are shown in Figs. 5 and 6, respectively. That they appear to have the same 'shape' agrees with the fact that each is just a scaling and shifting of $J(x^2 - \frac{3}{4})$.

4.2. *Graphs for which* $\beta = 2$, $n = 4$

For a graph G with independence number 2 on $n = 4$ vertices (and m non-edges), Eqs. (8) and (9) tell us that $f_G(x)$ is conjugate to $g_G(x) = x^2 - 2$ via $\phi(x) = mx + 2$, that is, $g_G = \phi \circ f_G \circ \phi^{\circ(-1)}$. Now $J(x^2 - 2)$ is well known (cf. [9, p. 226]) to be the interval $[-2, 2]$; applying the map $\phi^{\circ(-1)}(x) = \frac{1}{m}x - \frac{2}{m}$ to this interval gives

Theorem 4.3. *If G is a graph with independence number* 2 *having* $n = 4$ *vertices and m non-edges*, *then*

$$\mathscr{F}(G) = \left[\frac{-4}{m}, 0\right].$$

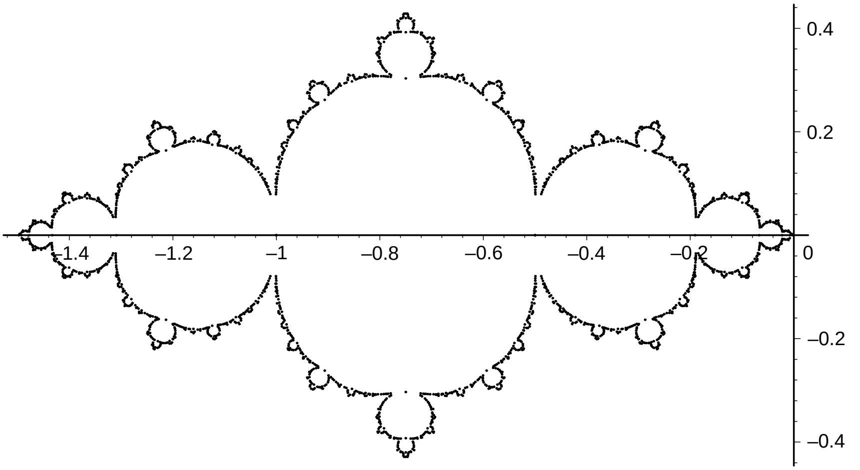

Fig. 5. The independence fractal, $\mathscr{F}(K_1 \uplus K_2)$.

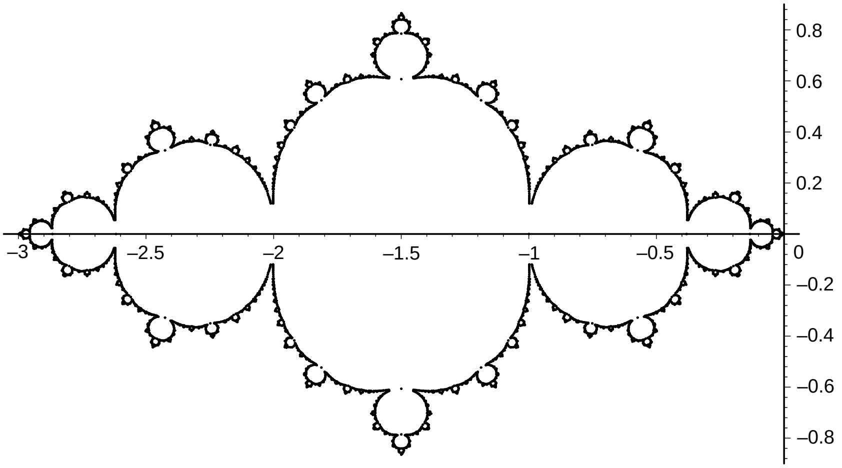

Fig. 6. The independence fractal, $\mathscr{F}(P_3)$.

For example, the graph $G = K_4 - e$ has independence number $\beta = 2$, $n = 4$ vertices and $m = 1$ non-edge. Then $f_G(x) = x^2 + 4x$ and, from Theorem 4.3, $\mathscr{F}(G) = [-4, 0]$.

4.3. *Graphs for which $\beta = 2$, $n \geqslant 5$*

If G is a graph with independence number 2 on $n \geqslant 5$ vertices, then $c = \frac{n}{2} - (\frac{n}{2})^2 < -2$, which lies outside the Mandelbröt set. This implies that $J(x^2 + c)$, and hence $\mathscr{F}(G) = \phi^{\circ(-1)}(J(x^2 + c))$, is fractal dust. Furthermore, it is known ([5, p. 126]) that for $c < -2$, $J(x^2 + c)$ is real and contained in the interval $[-q, q]$, where $q = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$. With $c = \frac{n}{2} - (\frac{n}{2})^2$, this simplifies to $q = \frac{n}{2}$. Applying $\phi^{\circ(-1)}$ to the interval $[-\frac{n}{2}, \frac{n}{2}]$ leaves $[-\frac{n}{m}, 0]$. We have proved:

Theorem 4.4. *If G is a graph with independence number 2 having $n \geqslant 5$ vertices and m non-edges, then $\mathscr{F}(G)$ is a dusty subset of the interval $[-\frac{n}{m}, 0]$.*

Since 0 is a repelling fixed point of f_G, it lies in $J(f_G)$. Furthermore, since $f(-\frac{n}{m}) = 0 \in J(f_G)$, and $J(f_G)$ is completely invariant under f_G (cf. Section 2), $-\frac{n}{m}$ also lies in $J(f_G)$. Hence, the interval $[-\frac{n}{m}, 0]$ in Theorem 4.4 is sharp.

Theorem 4.4 applies to the graph $K_2 \uplus K_3$, for example, which has independence number $\beta = 2$, $n = 5$ vertices and $m = 6$ non-edges. Here, $f_G(x) = 6x^2 + 5x$ and $\mathscr{F}(G)$ is a totally disconnected subset of the interval $[-\frac{5}{6}, 0]$.

5. The families ak_b and $k_{a;b}$

In this section, we make use of Theorem 3.7 to reveal the connection between the independence fractals of two seemingly different families of graphs with arbitrarily high independence numbers.

We consider first the graphs aK_b, the disjoint union of a copies of K_b. The independence fractals of $3K_2$ and $4K_2$ are shown in Figs. 7 and 8, respectively.

Note that $aK_b = \overline{K_a}[K_b]$; since $f_{\overline{K_a}}(x) = (1+x)^a - 1$ and $f_{K_b}(x) = bx$, we have

$$f_{aK_b}(x) = f_{\overline{K_a}}(bx) = (1+bx)^a - 1$$

and

$$f'_{aK_b}(x) = ab(1+bx)^{a-1},$$

whose only critical point is $z = -1/b$. By Theorem 3.5, $\mathscr{F}(G)$ will either be connected or totally disconnected, depending on whether the forward orbit of $z = -1/b$ is bounded or unbounded, respectively, in $(\mathbb{C}, |\cdot|)$.

Now, $f_{aK_b}(-1/b) = 0^a - 1 = -1$. As we are considering only non-empty graphs, $b \geqslant 2$.

Case 1: $b = 2$, a even. Then $f_{aK_b}(x) = f_{aK_2}(x) = (1+2x)^a - 1$. Now $f_{aK_2}(-1/b) = -1$, $f_{aK_2}(-1) = (1-2)^a - 1 = 0$, and $f_{aK_2}(0) = 0$. Hence, the forward orbit of $-1/b$ converges to 0, and is therefore bounded in $(\mathbb{C}, |\cdot|)$. Thus, $\mathscr{F}(aK_2)$ is connected.

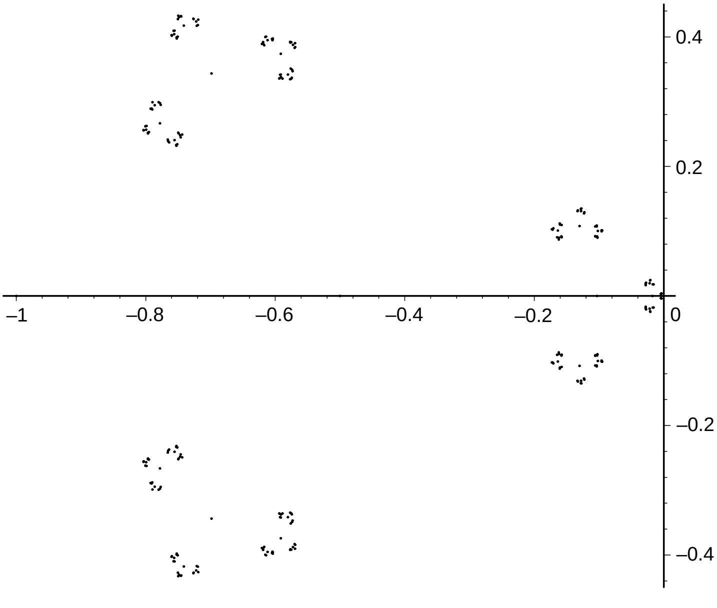

Fig. 7. The independence fractal, $\mathscr{F}(3K_2)$.

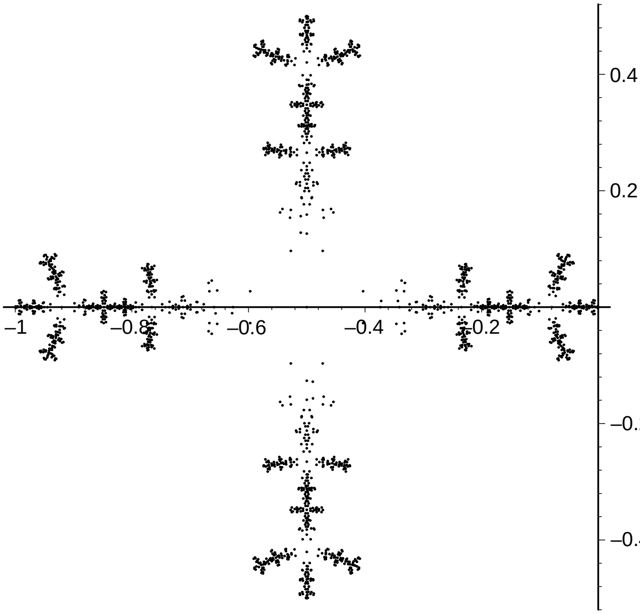

Fig. 8. The independence fractal, $\mathscr{F}(4K_2)$.

Case 2: $b \geqslant 3$, a even. Then $f_{aK_b}(-1/b) = -1$, and $f_{aK_b}(-1) = (1-b)^a - 1 \geqslant 2^a - 1 > 1$. And $z > 1 \Rightarrow f_{aK_b}(z) > (1+2z)^1 - 1 = 2z > z+1$. Hence, the forward orbit of $-1/b$ is unbounded in $(\mathbb{C}, |\cdot|)$, and $\mathscr{F}(aK_b)$ is totally disconnected.

Case 3: $a \geqslant 3$ odd. Then $f_{aK_b}(-1/b) = -1$, and $f_{aK_b}(-1) = (1-b)^a - 1 \leqslant (1-2)^3 - 1 = -2 < -1$. And $z < -1 \Rightarrow f(z) < (1+2z)^1 - 1 = 2z = z + z < z - 1$. Hence, the forward orbit of $-1/b$ is unbounded in $(\mathbb{C}, |\cdot|)$, and $\mathscr{F}(aK_b)$ is totally disconnected.

Case 4: $a = 1$. Then $aK_b = K_b$, whose independence fractal we know is $\{0\}$, and thus totally disconnected.

We have proved:

Theorem 5.1. *The independence fractal of aK_b is connected if $b = 2$ and a is even, and totally disconnected otherwise.*

As we did for graphs with independence number 2, we can find a region inside which $\mathscr{F}(aK_b)$ lies. It lies in the disk

$$\left| z + \frac{1}{b} \right| \leqslant \frac{1}{b},$$

and this is a direct consequence of Theorem 5.2.

Theorem 5.2. *For $G = aK_b$ and all $k \geqslant 1$, every root F_k of f_{G^k} satisfies $|F_k + \frac{1}{b}| \leqslant \frac{1}{b}$.*

Proof. By induction on k. Since $f_G(x) = (1+bx)^a - 1$, every root F_1 of $f_G(x)$ satisfies $(1+bF_1)^a = 1$, which implies that $|1+bF_1| = 1$ and thus $|F_1 + \frac{1}{b}| = \frac{1}{b}$; the result is therefore true for $k = 1$.

Now suppose the result is true for a number $k \geqslant 1$. Since $f_{G^{k+1}}(x) = f_{G^k}(f_G(x))$, any root F_{k+1} of $f_{G^{k+1}}(x)$ must satisfy $f_G(F_{k+1}) = F_k$ for some root F_k of $f_{G^k}(x)$. This says that $(1+bF_{k+1})^a - 1 = F_k$, and so $|1+bF_{k+1}| = |1+F_k|^{1/a}$ and $|F_{k+1} + \frac{1}{b}| = \frac{1}{b}|1 + F_k|^{1/a}$. By assumption, $|F_k + \frac{1}{b}| \leqslant \frac{1}{b}$. Applying the triangle inequality,

$$\begin{aligned}\left|F_k + \frac{1}{b}\right| &= \frac{1}{b}|1+F_{k-1}|^{1/a}\\ &= \frac{1}{b}\left|\left(F_{k-1} + \frac{1}{b}\right) + \left(1 - \frac{1}{b}\right)\right|^{1/a}\\ &\leqslant \frac{1}{b}\left(\left|F_{k-1} + \frac{1}{b}\right| + \left|1 - \frac{1}{b}\right|\right)^{1/a}\\ &\leqslant \frac{1}{b}\left(\frac{1}{b} + \left(1 - \frac{1}{b}\right)\right)^{1/a}\\ &= \frac{1}{b},\end{aligned}$$

and so the result holds for $k+1$ as well, completing the proof. $\square$

The bounding disk $|z + \frac{1}{b}| \leqslant \frac{1}{b}$ is best possible, since (as revealed in the proof of Theorem 5.2) the roots of f_{aK_b}, itself, lie on the boundary.

Consider next the family of complete multipartite graphs $K_{a;b} = K_{\underbrace{a, a, \ldots, a}_{b}}$. Here, again $b \geqslant 2$ since we are considering only non-empty graphs. Since $K_{a;b}$ is precisely the graph $K_b[\overline{K_a}]$, we have

$$f_{K_{a;b}}(x) = f_{K_b[\overline{K_a}]}(x) = b.$$

$$f_{\overline{K_a}}(x) = b(1+x)^a - b.$$

Moreover, since $K_{a;b} = K_b[\overline{K_a}]$ and $aK_b = \overline{K_a}[K_b]$, Theorem 3.7 tells us that

$$\mathscr{F}(K_{a;b}) = b.\ \mathscr{F}(aK_b). \tag{11}$$

In particular, $\mathscr{F}(K_{a;b})$ has the same 'shape' as its counterpart, $\mathscr{F}(aK_b)$. Together with Theorems 5.1 and 5.2, Eq. (11) implies:

Theorem 5.3. *The independence fractal of $K_{a;b}$ is connected if $b = 2$ and a is even, and totally disconnected otherwise. Further, $\mathscr{F}(K_{a;b})$ lies in the disk $|z+1| \leqslant 1$, and this bounding disk is best possible.*

6. Concluding remarks

The relationships between a graph and its independence fractal remains a tantalizing question. Even the restricted question of when an independence fractal is connected seems elusive. Certainly, it does not depend on the connectivity of the graph. We have seen, for instance, that $4K_2$, a disconnected graph, has a connected independence fractal, while Theorem 3.8 guarantees the existence of many connected graphs with disconnected independence fractals. In Section 4, we were able to provide a complete answer for graphs with independence number 2, and it may be possible to do something similar for graphs with independence number 3, though the Mandelbröt set for cubics is contained in $\mathbb{C} \times \mathbb{C}$ [3,4] and is not well understood.

Just how much about a graph can its independence fractal tell us? Theorem 3.7 tells us that $G[K_n]$ and $K_n[G]$ have analytically conjugate independence fractals. Further, since for any polynomial f and positive integer k, $J(f) = J(f^{\circ k})$ (Theorem 2.4), it follows that G^k and G have identical independence fractals for any graph G. These observations provide a partial answer to:

Question 6.1. *When do two graphs G and H have analytically conjugate independence fractals?*

Finally, related to the problem of determining bounds for the roots of independence polynomials [6,8] is that of bounding independence fractals in terms of various graph parameters. Theorems 5.2 and 5.3 tell us that $\mathscr{F}(aK_b)$ and $\mathscr{F}(K_{a;b})$ lie in the disks $|z + \frac{1}{b}| \leqslant \frac{1}{b}$ and $|z + 1| \leqslant 1$, respectively, while Theorem 4.1 implies that for graphs with independence number 2 the independence fractals lie in $|z + \frac{n}{2m}| \leqslant \frac{n}{2m}$. It is not clear what a general result along these lines might be.

Appendix A. Proof of Theorem 2.3

We shall make use of three results in the literature.

Theorem A.1. (cf. Beardon [1, p. 71]). *Let f be a rational map of degree at least two, and E a compact subset of $\mathbb{C}_\infty$ such that for all $z \in F(f)$, the sequence $\{f^{\circ k}(z)\}$ does not accumulate at any point of E. Then for any open set U containing $J(f)$, $f^{\circ(-k)}(E) \subseteq U$ for all sufficiently large k.*

Theorem A.2. (cf. Beardon [1, p. 149]). *Let f be a rational map of degree at least two, W a domain that meets J, and K any compact set containing no exceptional points of f. Then for all sufficiently large k, $f^{\circ k}(W) \supset K$.*

(A domain is an open connected set.)

Recall from Section 2.1 the definition of irrationally indifferent cycles. If z_0 lies on an irrationally indifferent cycle of f with period k, and this cycle lies in the Fatou set

$F(f)$, then the component (i.e., maximal open connected subset) F_0 of $F(f)$ containing z_0 is forward invariant under $f^{\circ k}$, and is called a *Siegel disk*. For any point $z \neq z_0$ in F_0, the sequence $\{f^{\circ k}(z), f^{\circ(2k)}(z), \dots\}$ is dense on a curve—called, an *invariant circle*—lying inside F_0.

In fact, a complete classification of the possibilities for periodic components of a rational map is known; and every component C of a Fatou set $F(f)$ is eventually periodic under f, that is, for some $j > k \geqslant 0$, $f^{\circ j}(C) = f^{\circ k}(C)$. These very deep and fascinating results were proved by Sullivan (cf. [1] for references and details), and an immediate consequence of his work is:

Theorem A.3. *If f is a polynomial, and $z_0 \in F(f)$, then the forward orbit $\mathcal{O}^+(z_0) = \{f^{\circ k}(z_0)\}$ either*

(i) *converges to a periodic cycle, or*
(ii) *settles into a 'periodic cycle' of Siegel disks, becoming dense on an invariant circle in each.*

Unbounded forward orbits actually converge to the point at infinity with respect to the spherical metric, σ_0. This situation is covered by (i) in Theorem A.3, since ∞ is a fixed (and thus, periodic) point of any polynomial.

With these results at hand, we can prove Theorem 2.3.

Proof of Theorem 2.3. Let f and z_0 be as in the statement of the theorem, and $\varepsilon > 0$ given. Establishing the limit in the conclusion of the theorem is equivalent (cf. [1, p. 35]) to proving that, for all sufficiently large k,

(i) $f^{\circ(-k)}(z_0) \subseteq J(f) + \varepsilon$, and
(ii) $J(f) \subseteq f^{\circ(-k)}(z_0) + \varepsilon$,

where $A + \varepsilon = \{z : \sigma_0(z, a) < \varepsilon \text{ for some } a \in A\}$, the dilation of A by a ball of radius ε.

To prove (i), note first that if $z_0 \in J(f)$ then $f^{\circ(-k)}(z_0) \subseteq J(f) \subseteq J(f) + \varepsilon$ for all k. Assume, then, that $z_0 \in F(f)$. From Section 2.1, the periodic cycles in $F(f)$ are either attracting or irrationally indifferent, the latter lying in Siegel disks. Thus, since z_0 lies in neither an attracting cycle nor a Siegel disk, Theorem A.3 implies that no point z in $F(f)$ will have a forward orbit that accumulates at z_0. Hence, the set $E = \{z_0\}$ satisfies the hypothesis of Theorem A.1, and therefore $f^{\circ(-k)}(z_0) \subseteq J(f) + \varepsilon$ for all sufficiently large k.

To prove (ii), we begin by choosing a positive number $\delta < \varepsilon/2$, and covering $J(f)$ with finitely many open balls of radius δ (such a covering exists since $J(f)$ is compact). The point z_0 is not exceptional, since exceptional points are necessarily periodic points in $F(f)$. For each ball W in the covering $\mathscr{W}$, Theorem A.2 implies that for all sufficiently large k, $f^{\circ(k)}(W) \supset \{z_0\}$, and hence $f^{\circ(-k)}(z_0) \cap W \neq 0$. Since there are only finitely many such balls, we then have that, for all sufficiently large

k, $f^{\circ(-k)}(z_0) \cap W \neq \emptyset$ for each ball W. Finally, since $\varepsilon > 2\delta$ it follows that for all such k, $f^{\circ(-k)}(z_0) + \varepsilon \supset \mathscr{W} \supset J(f)$.

This completes the proof. $\square$

Appendix B. A tight box containing $J(x^2 - \frac{3}{4})$

We give a proof of the claim in Section 4 that the Julia set of $g(x) = x^2 - \frac{3}{4}$ is contained in the box $[-\frac{3}{2}, \frac{3}{2}] \times [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$. This box is in fact best possible, as the points $\pm\frac{3}{2}$ and $+\frac{\sqrt{3}}{2}i$ all lie in $J(g(x))$: the point $z = \frac{3}{2}$ is a repelling fixed point of g, and $g^{\circ 2}(\pm\frac{\sqrt{3}}{2}i) = g(-\frac{3}{2}) = \frac{3}{2} \in J(g(x))$. Since repelling fixed points of g lie $J(g(x))$, and $J(g)$ is completely invariant (cf. Section 2), it follows that $\pm\frac{\sqrt{3}}{2}$ and $\pm\frac{\sqrt{3}}{2}i$ all lie in $J(g(x))$.

Lemma B.1. *For $g(x) = x^2 - \frac{3}{4}$ and $z = a + bi \in \mathbb{C}$, if either $|a| > \frac{3}{2}$ or $|b| > \frac{\sqrt{3}}{2}$ then $|g(z)| > \frac{3}{2}$.*

Proof. We have

$$g(z) = z^2 - \tfrac{3}{4} - (a^2 - b^2 - \tfrac{3}{4}) + 2abi$$

and so

$$|g(z)|^2 = a^4 + b^4 + 2a^2b^2 + \tfrac{3}{2}b^2 - \tfrac{3}{2}a^2 + \tfrac{9}{16}. \tag{B.1}$$

Now, if $|b| > \frac{\sqrt{3}}{2}$ then, from (A.1),

$$\begin{aligned} |g(z)|^2 &\geqslant b^4 + 2a^2b^2 + \tfrac{3}{2}b^2 - \tfrac{3}{2}a^2 + \tfrac{9}{16} \\ &= (2b^2 - \tfrac{3}{2})a^2 + b^4 + \tfrac{3}{2}b^2 + \tfrac{9}{16}) \\ &> (2 \cdot \tfrac{3}{4} - \tfrac{3}{2})a^2 + \tfrac{9}{16} + \tfrac{3}{2} \cdot \tfrac{3}{4} + \tfrac{9}{16} \\ &= \tfrac{9}{4}, \end{aligned}$$

and hence $|g(z)| > \frac{3}{2}$.

On the other hand, if $|a| > \frac{3}{2}$ then, from (A.1),

$$\begin{aligned} |g(z)|^2 &\geqslant a^4 - \tfrac{3}{2}a^2 + \tfrac{9}{16} \\ &= a^2(a^2 - \tfrac{3}{2}) + \tfrac{9}{16} \\ &> \tfrac{9}{4}(\tfrac{9}{4} - \tfrac{3}{2}) + \tfrac{9}{16} \\ &= \tfrac{9}{4}, \end{aligned}$$

which also implies that $|g(z)| > \frac{3}{2}$. $\square$

Next, we prove:

Lemma B.2. *For $g(x) = x^2 - \frac{3}{4}$, if $z \in \mathbb{C}$ is such that $|z| \geqslant \frac{3}{2} + \epsilon$ for some $\epsilon > 0$, then $|g(z)| \geqslant \frac{3}{2} + 3\epsilon$.*

Proof. We have

$$\begin{aligned} |g(z)| &= |z^2 - \tfrac{3}{4}| \\ &\geqslant |z|^2 - \tfrac{3}{4} \\ &\geqslant (\tfrac{3}{2} + \epsilon)^2 - \tfrac{3}{4} \\ &= \tfrac{3}{2} + 3\epsilon + \epsilon^2 \\ &> \tfrac{3}{2} + 3\epsilon. \end{aligned}$$

Together, Lemmas A.1 and A.2 imply:

Theorem B.3. *For $g(x) = x^2 - \frac{3}{4}$ and $z \in \mathbb{C}$, if either $|\mathscr{R}e\,(z)| > \frac{3}{2}$ or $|\mathscr{I}m\,(z)| > \frac{\sqrt{3}}{2}$ then $|g^{\circ k}(z)| \to \infty$ as $k \to \infty$.*

Proof. As z satisfies the hypothesis of Lemma A.1, we have $|g(z)| = \frac{3}{2} + \epsilon$ for some $\epsilon > 0$. Applying Lemma A.2 to $g(z)$, we have $|g^{\circ 2}(z)| \geqslant \frac{3}{2} + 3\epsilon$, and, by induction, we find that $|g^{\circ k+1}(z)| \geqslant \frac{3}{2} + 3^k\epsilon$ for each $k \geqslant 1$, and the conclusion of theorem follows. □

Theorem B.3 implies that the Filled Julia set (and hence the Julia set) of $g(x) = x^2 - \frac{3}{4}$ lies in the box $[-\frac{3}{2}, \frac{3}{2}] \times [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$.

Corollary B.4. *The Julia set $J(x^2 - \frac{3}{4})$ is contained in the box*

$$\left\{ z : |\mathscr{R}e\,(z)| \leqslant \frac{3}{2} \quad \text{and} \quad |\mathscr{I}m\,(z)| \leqslant \frac{\sqrt{3}}{2} \right\}.$$

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