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Extending precolorings of subgraphs of locally planar graphs

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Abstract

The width of an embedded graph is the length of a shortest noncontractible cycle. Suppose G is embedded in a surface S (either orientable or not) with large width. In this case G is said to be locally planar. Suppose $P \subset V(G)$ is a set of vertices such that the components of $G[P]$ are each 2-colorable, have bounded diameter and are suitably distant from each other. We show that any 5-coloring of $G[P]$ in which each component is 2-colored extends to a 5-coloring of all of G. Thus, for an arbitrary surface, the extension theorems for precolorings of subgraphs of locally planar graphs parallel the results for planar graphs. Crucial to the proof of this result is the nice cycle lemma, viz. If C is a minimal, noncontractible, and nonseparating cycle in a so-called *orderly* triangulation of at least moderate width, then there is a cycle C' such that C' lies within the fourth neighborhood of C, C' is minimal, homotopic to C, and C' either has even length or contains a vertex of degree 4. Such a *nice* cycle is useful in producing 5-colorings. We introduce the idea of *optimal shortcuts* in order to prove the nice cycle lemma and the idea of *relative width* in order to prove the main theorem. Our results generalize to extension theorems for precolorings with $q \geq 3$ colors.
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1. Introduction

Suppose G is an r-colorable graph and $P \subseteq V(=V(G))$ is r-precolored. It is natural to ask whether the precoloring extends to a t-coloring of the entire graph where $r \leq t$ [11, 12]. The answer depends on the context. For example, a 4-precoloring of some vertices of a planar graph need not extend to a 4-coloring of the entire graph even if there are only two

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precolored vertices and they are far apart. In contrast, a 5-precoloring of suitably separated vertices extends to a 5-coloring of any planar graph [1].

It is no surprise that the question of whether a precoloring extends is NP-complete even assuming severe restrictions on G. Its complexity has been well studied; see [14]. Still there are general results. We introduce a little notation before describing some of what is known.

The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to properly color the vertices of G. We let $G[P]$ denote the subgraph of G induced by P. P is called *independent* if $G[P]$ contains no edges. It is convenient to partition $P = P_1 \cup P_2 \cup \cdots \cup P_k \subseteq V$ where $P_1, P_2, \ldots, P_k$ induce the connected components of $G[P]$. We say that $d(P) \geq \rho$ if for every pair of vertices $x \in P_i, y \in P_j$ $(i \neq j)$ the distance (number of edges in a shortest path) between x and y is at least ρ. The story begins with a precolored independent set.

Theorem 1 ([1]). *If $\chi(G) = r$, P is independent, and $d(P) \geq 4$, then any $(r+1)$-coloring of $G[P]$ extends to an $(r+1)$-coloring of all of G.*

This generalizes to extending precolorings of q-colorable subgraphs.

Theorem 2 ([3]). *Suppose $\chi(G) \leq r$ and $P \subset V$ is such that $\chi(G[P]) \leq q$ and $d(P) \geq 4$. Any $(r+q)$-coloring of $G[P]$ in which each $G[P_i]$ is q-colored (not necessarily with the same colors) extends to an $(r+q)$-coloring of G.*

The preceding results are best possible both with respect to the distance constraint and the number of colors necessary. If we assume that G is planar, we can save a color. Note that the following theorem is not true for any distance constraint when $q = 1$ [1].

Theorem 3 ([3]). *Suppose G has no K_5 minor, $\chi(G) \leq r$, and $P \subset V$ is such that $d(P) \geq 8$ and $\chi(G[P]) \leq q$ where $q \geq 2$. Any $(r+q-1)$-coloring of $G[P]$ in which each $G[P_i]$ is q-colored (not necessarily with the same colors) extends to an $(r+q-1)$-coloring of G.*

It is natural to wonder if Theorem 3 extends to the class of "locally planar" graphs. Our principal result is that this is indeed so.

For a surface S, let $\epsilon = \epsilon(S)$ denote its Euler characteristic, and let the *Euler genus* be given by $g^* = 2 - \epsilon$. When G is embedded in a nonplanar surface, its *width* $w(G)$ is the length of a shortest noncontractible cycle [4]. This has also been called the *edge-width* and is known to be a crucial embedding parameter; see [8, Chapter 5]. A graph G embedded in a surface S is said to be *locally planar* if $w(G)$ is large enough. Here "large enough" will depend on S as well as the property being sought. Theorem 3 and Thomassen's landmark Theorem 4 provide the origin for our work. The latter also illustrates the importance of width.

Theorem 4 ([13]). *Let G be embedded in S. If $w(G)$ is large enough, then G can be 5-colored.*

We will prove two 5-coloring extension theorems for locally planar graphs. Within the introduction we hide details. Our first result, assuming a precolored independent set, is stated and proved in Section 3; the second, assuming a precolored bipartite subgraph, fills Section 4.

Theorem 5. *Let G be embedded in a surface S of Euler genus $g^* > 0$ such that $w(G)$ is large enough. If $P \subset V$ is an independent set such that $d(P)$ is large enough, then any 5-coloring of $G[P]$ extends to a 5-coloring of G.*

Ideally we would like a 5-coloring extension theorem when $G[P]$ is q-colored for $q < 5$ and $d(P)$ is suitably bounded. If $q \geq 3$, there is no such theorem [3]. To obtain a result for $q = 2$ we need to control the embedding a bit more. For $P \subset V$ we introduce the *relative width* of G, denoted by $w_P(G)$. This is the length of a shortest noncontractible cycle in which the vertices of P do not count. Formally $w_P(G) = \min\{|X| : X \subset V$ and $G[X \cup P]$ contains a noncontractible cycle$\}$.

Theorem 6. *Suppose G is embedded in a surface S of Euler genus $g^* > 0$ and $P \subset V$ is such that for each component the diameter of $G[P_i]$ is bounded. If both $d(P)$ and $w_P(G)$ are large enough, then any 5-coloring of $G[P]$ in which each component is 2-colored (not necessarily with the same colors) extends to a 5-coloring of G.*

It is immediate that $0 \leq w_P(G) \leq w(G)$. The width of an embedded graph can be determined in polynomial time [8]. Determining the relative width is the same as determining the width of the graph obtained from G by contracting each component of $G[P]$. If $G[P]$ is noncontractible on the surface, the Euler genus of the contracted graph will decrease (with the graph possibly now on a pinched surface). Since this is easy to detect both the width and the relative width are accessible parameters.

Both of our precoloring extension theorems require the nice cycle lemma. Although this has become part of the folklore of topological graph theory, it seems tricky to prove. To the best of our knowledge the proof we give below is the first published argument that works for surfaces with $g^* > 2$. We devote Section 2 to developing this important technical result and introduce the idea of "optimal k-shortcuts" for our proof.

We anticipate that both relative width and optimal shortcut will have further application in topological graph theory.

2. The nice cycle lemma

The driving force behind our results and many of their predecessors is the *nice cycle lemma*. This says that if C is a noncontractible cycle in an "orderly" triangulation (defined below) then there is a minimal cycle C' that is near to C, homotopic to C, and particularly nice from the point of view of 5-coloring. A minimal (i.e., chordless) noncontractible, and nonseparating cycle C in an embedded graph is said to be *nice* if it either has even length or contains a vertex of degree at most 4. The first published version of a nice cycle lemma appeared in [5]. There the cycle C had to be a minimum length noncontractible cycle in an orderly triangulation of the torus, and the cycle C' was found in the first neighborhood of C. Stromquist established a nice cycle lemma for an arbitrary surface where C' was found in the "filled first neighborhood" of C [10], but this was never published. The filled first neighborhood might contain vertices arbitrarily far from C. Thomassen constructed a nice cycle in [13]; however, to obtain his version of C' he used the notion of weak geodesic, requiring a considerable detour away from C. Using local modifications he found his nice cycle on one side of C, but, as we now know, finding a nice cycle requires zigzagging to both sides of C. We sketch the argument at the end of this section.

Here we introduce optimal k-shortcuts and use this idea to give a simple proof of the lemma. Our result also has the advantage that C' is within distance four of the original C. We begin with some necessary background. If $U \subset V$, let $N(U)$ denote the set of vertices that are not in U but are adjacent to at least one vertex in U. Inductively $N^i(U)$ denotes the set of vertices that are not in $N^{i-1}(U)$ but are adjacent to at least one vertex in $N^{i-1}(U)$. Thus $N^i(U)$ consists of those vertices in G whose distance from U is exactly i.

Given a traversal of C, a 2-sided cycle within a graph G embedded in a surface S, we define $R(C)$ and $L(C)$, respectively the (not necessarily disjoint) right and left neighbors of C. $R(C)$ (resp., $L(C)$) is the set of vertices of $N(C)$ that are adjacent to C along an edge emanating from the right (resp., left) side of C. All separating cycles of G are 2-sided as are all noncontractible cycles on an orientable surface. A nonorientable surface may contain 1-sided or 2-sided noncontractible and nonseparating cycles.

For G embedded in a surface S of Euler genus $g^* > 0$, a set of noncontractible cycles $C_1, C_2, \ldots, C_s$ where $s \leq g^*$ is called a *planarizing set* if $G - \cup C_i$ is a planar graph. For $i \neq j$, $\text{dist}(C_i, C_j)$ is the length of a shortest path joining these cycles. When C_i is a 2-sided noncontractible and nonseparating cycle, $\text{dist}(C_i, C_i)$ is the length of a shortest path beginning with an edge from C_i to $L(C_i)$ and ending with an edge from $R(C_i)$ to C_i.

When C_i is a 1-sided noncontractible cycle, it is convenient to define $\text{dist}(C_i, C_i)$ as infinity. In this case we (locally) define R and L along M, a subpath of C_i that contains m edges. Suppose C_i is a (1-sided or 2-sided) noncontractible cycle of length at least $m + 1$. On a traversal of $M \subset C_i$ there are right neighbors $R(M)$ and left neighbors $L(M)$, both subsets of $N(C_i)$, and these two sets are disjoint when $m \leq w(G) - 3$. Formally when $m \leq w(G) - 3$, $R(M) = \{y \in N(C_i)$: there is a vertex $x \in M$ and an edge xy lying on the right side of M in the traversal$\}$. $L(M)$ is defined in the same manner and given the width hypothesis, $L(M) \cap R(M) = \emptyset$.

We define $R^j(M)$ to be the vertices of $N^j(C_i)$ that can be reached by a path starting in M with an edge of $R(M)$. We define $L^j(M)$ in an analogous way and note that for $2 \leq j \leq \lfloor \frac{w(G)-m-1}{2} \rfloor$, $L^j(M) \cap R^j(M) = \emptyset$. For a vertex $x \in C_i$ we define $R(x)$ to be its right neighbors with respect to a traversal of a (short) subpath containing x and $\deg_R(x) = |R(x)|$ is called the *right degree* of x. We make the analogous definition for the *left degree* and note that $\deg(x) = \deg_R(x) + \deg_L(x) + 2$.

If G is a triangulation of the surface S and $U \subset V$, $G[U]$ is said to be *orderly* if every contractible 3-cycle in $G[U]$ bounds a face of G and if every contractible 4-cycle in $G[U]$ is either the boundary of two triangles of G that share an edge or the first neighbor circuit of a vertex of degree 4 in G. Orderly graphs have been helpful in inductive proofs of 5-coloring theorems for embedded graphs [5, 7].

Suppose $C_1 = u_1, u_2, \ldots, u_t$ is a minimal, noncontractible, and nonseparating cycle in a graph embedded on a surface S of Euler genus $g^* > 0$. A path of length 6, say $v_1, v_2, \ldots, v_7$ is said to be an *optimal* 6-*shortcut* for C_1 if $v_1 = u_i$ for some i, $v_7 = u_j$ for some j, the resulting cycle $C_2 = u_1, \ldots, u_{i-1}, \{u_i = v_1\}, v_2, \ldots, v_6, \{v_7 = u_j\}, u_{j+1}, \ldots, u_t$ is homotopic to C_1, **and** C_2 is as short as possible. Note that $v_2, \ldots, v_6$ may be vertices of C_1.

Lemma 1 (Nice Cycle Lemma). *Suppose C_1 is a minimal, noncontractible, and nonseparating cycle in a graph G, a triangulation of a surface S of Euler genus $g^* > 0$. Suppose $G_{C_1,4} = G[C_1 \cup N(C_1) \cup \cdots \cup N^4(C_1)]$ is orderly and $w(G) \geq 15$. There is a minimal, noncontractible, and nonseparating cycle, say C_1', in $G_{C_1,4}$ such that C_1' is homotopic to C_1 and C_1' is nice. If $\text{dist}(C_1, C_1) \geq d$, then $\text{dist}(C_1', C_1') \geq d - 8$, and for $X \subset V$ disjoint from C_1, $\text{dist}(X, C_1') \geq \text{dist}(X, C_1) - 4$.*

Proof. Suppose $C_1 = u_1, u_2, \ldots, u_t$ is a minimal, noncontractible, and nonseparating cycle in G. Choose $M = v_1, v_2, \ldots, v_7$ to be an optimal 6-shortcut for C_1 where $v_1 = u_i$ and $v_7 = u_j$. The resulting cycle $C_2 = u_1, \ldots, \{u_i = v_1\}, v_2, \ldots, v_6, \{v_7 = u_j\}, u_{j+1}, \ldots, u_t$ is minimal since if C_2 were to contain a chord then the 6-shortcut would not have been optimal. Since $w(G) \geq 15$, $R^4(M) \cap L^4(M) = \emptyset$. We may assume that both C_1 and C_2 have odd length and contain no vertex of degree 4. Thus both t and $i - 1 + 7 + t - j$ are odd. If D denotes the path from u_{j+1} to u_{i-1} inclusive, then $|D| = i - 1 + t - j$ is even.

Case (i). Suppose $\deg_R(v_3) \geq 2$ and $\deg_R(v_4) \geq 2$. Let x denote the vertex that is in a triangle to the right of the edge joining v_3 with v_4. Suppose A is a shortest path from v_1 to x among the vertices in $N(v_3)$ not including either endpoint. Normally A includes v_2 but that is not necessary. Next let B be the shortest path from x to v_6 among the neighbors of v_4 not including either endpoint. Normally B includes v_5 but that is not necessary.

Consider the cycle $C_3 = D, v_1, A, x, v_4, v_5, v_6, v_7$. C_3 is homotopic to C_2. Let us examine the possible chords for C_3. If there were a chord from D to any vertex in the subpath A, x, v_4, v_5, v_6, then we did not select our optimal 6-shortcut correctly. This is also the case if there were a chord from A to either v_6 or v_7. If there were a chord from A to v_4, then $G_{C_1,4}$ contains a contractible 3-cycle that is not a face boundary. If there were a chord from A to v_5, then we have a contractible 4-cycle, say a, v_3, v_4, v_5, with x in its interior. If x is adjacent to v_5 and $\deg(x) = 4$, then $D, v_1, v_2, v_3, x, v_5, v_6, v_7$ is nice. If x is not adjacent to v_5 or $\deg(x) \neq 4$, then $G_{C_1,4}$ contains a forbidden contractible 4-cycle. Thus either $|A|$ is odd or we are done since C_3 would be nice.

Next consider the cycle $C_4 = D, v_1, v_2, v_3, x, B, v_6, v_7$. As in the preceding paragraph either $|B|$ is odd or we are done since C_4 would be nice. Finally, look at $C_5 = D, v_1, A, x, B, v_6, v_7$. C_5 is homotopic to C_2 and $|C_5|$ is even. The only chord we have not considered is an edge from A to B. This will produce a contractible 4-cycle. Either $\deg(x) = 4$ and C_4 is nice or $G_{C_1,4}$ contains a forbidden contractible 4-cycle. Thus we are finished the proof of Case (i).

Case (ii). Suppose C_2 is our minimal, noncontractible, and nonseparating cycle and $\deg_R(v_3) < 2$. If either $\deg_R(v_3) = 0$ or $\deg_R(v_3) = \deg_R(v_4) = 1$, then $v_1, \ldots, v_7$ is not an optimal 6-shortcut. Thus we may assume that $\deg_R(v_3) = 1$ and $\deg_R(v_4) \geq 2$. If $\deg_R(v_5) \geq 2$, then we are back in Case (i) using v_4 and v_5. Thus we may assume $\deg_R(v_5) = 1$. If either $\deg_L(v_5) = 1$ or $\deg_L(v_3) = 1$, then C_2 is nice. Now if $\deg_L(v_4) > 1$, then we are back in Case (i) interchanging left with right. Thus we may assume that $\deg_R(v_3) = \deg_L(v_4) = 1$ and $\deg_L(v_5) \geq 2$. Suppose x is v_3's unique right neighbor. We alter C_2 by replacing v_3 with x. In the resulting cycle both $\deg_L(v_4)$ and $\deg_L(v_5)$ are at least 2 and we are in Case (i) once again.

Case (iii). Suppose C_2 is our minimal, noncontractible, and nonseparating cycle, $\deg_R(v_3) \geq 2$, and $\deg_R(v_4) < 2$. As in Case (ii) we may assume $\deg_R(v_4) = 1$. If $\deg_R(v_5) \leq 1$, then $v_1, \ldots, v_7$ is not an optimal 6-shortcut. If $\deg_R(v_5) \geq 2$, then by reversing the direction of the 6-shortcut we are in Case (ii).

In each case we find the nice cycle within $G_{C_1,4}$. Thus each nice cycle is no more than four edges away from its original planarizing cycle. The distance claims follow immediately. □

We digress a moment to describe how to construct a noncontractible cycle C in a graph embedded in a nonorientable surface with the property that a nearby nice cycle must use both sides of C. Let C consist of vertices whose right degrees are alternately 1 and 3. It is easy to build such a cycle in a 6-regular triangulation of a Klein bottle, and from there it can be on any nonorientable surface. For such a cycle, there is no nice replacement cycle lying locally on one side of C. Thus when C has odd length and is 1-sided, the nice cycle must be found using detours locally to both sides of C.

3. When an independent set is precolored

A version of the next theorem for orientable surfaces appeared in [2]. There are several good reasons to have another look. First, the proof given below works for both orientable and nonorientable surfaces. Second, the particular nice cycle lemma needed in the earlier proof never appeared in print. Third, several steps in our proof are reused in Section 4. Finally, the constants are better. Note that Theorem 7 is a specific realization of Theorem 5.

Theorem 7. *Suppose G is embedded in a surface S of Euler genus $g^* > 0$ and $w(G) \geq 208(2^{g^*} - 1)$. If $P \subset V$ is an independent set in G such that $d(P) \geq 18$, then any 5-coloring of P extends to a 5-coloring of G.*

Proof. We may assume G is a triangulation of S. Otherwise we could add vertices and edges to G, making it a triangulation while keeping the width and distance between vertices unchanged [2, 5, 12]. Thomassen and Yu have shown that if $w(G) \geq 8(d+1)(2^{g^*}-1)$, then there exists a planarizing set of cycles $C_1, C_2, \ldots, C_s$, $s \leq g^*$, such that $\text{dist}(C_i, C_j) \geq 2d$ for $1 \leq i \leq j \leq s$ [13, 15]. For our purposes we need $d = 25$. If for some $p \in P$ and $i \leq s$, $\text{dist}(p, C_i) < 9$, we create a new minimal cycle, say C_i', by replacing vertices whose distance is less than 9 from p by a path in $N^9(p)$. With the given distance constraints, these altered C_i' do not intersect and do not come too close to another $p^* \in P$.

Once we have done this for every cycle in $\{C_1, \ldots, C_s\}$ and for every $p \in P$, we have a new collection of planarizing cycles $\{C_1', \ldots, C_s'\}$ with each C_i' homotopic to C_i. Since the maximum distance between any two vertices within $N^9(p)$ is 18, the distance involving one cycle can be changed by at most 18. Since $\text{dist}(C_i, C_j) \geq 2d = 50$, we have $\text{dist}(C_i', C_j') \geq 2d - 36 = 14$ and $\text{dist}(P, C_i') \geq 9$ for each $i, j \leq s$.

Next, for each $i \leq s$ we make the subgraph $G_{C_i',4}$ orderly (see Lemma 1). For each $G_{C_i',4}$ we delete vertices of G interior to any contractible 3-cycle of $G_{C_i',4}$ and, if there is more than one, all vertices interior to any contractible 4-cycle of $G_{C_i',4}$. In the latter case, we add a vertex v^* adjacent to all four boundary vertices. We may have deleted vertices of P, but as we shall see below, none interior to and adjacent to vertices of a contractible

3- or 4-cycle. In addition we have not changed the distance between any two vertices of P exterior to these contractible cycles. Thus it is still the case that $d(P) \geq 18$ and $\text{dist}(P, C'_i) \geq 9$. Call this intermediate graph G_1. Then for $1 \leq i \leq s$ we apply Lemma 1 with $C'_i = C_1$ and let C''_i denote the resulting nice cycle, homotopic to C'_i, lying within $G_{C'_i,4}$. This gives the final planarizing set of cycles $\{C''_1, \ldots, C''_s\}$.

We have $\text{dist}(C''_i, C''_j) \geq 2d - 36 - 8 = 6$ and $\text{dist}(P, C''_i) \geq 5$. With this planarizing set for G_1 we form a planar triangulation. For each C''_i that is 2-sided, $1 \leq i \leq s$, we remove C''_i and replace it with two vertices x_i and y_i. Then for each edge in $L(C''_i)$, x_i is adjacent to both endpoints of that edge, and for each edge in $R(C''_i)$, y_i is adjacent to both endpoints. For C''_i 1-sided, we remove C''_i and replace it with x_i adjacent to both ends of each edge in $N(C''_i)$. The resulting graph G_0 is a plane triangulation with the distance between every pair of new vertices, x_i, y_i, at least 6. Their distance from P is at least 5.

Suppose c is a 4-coloring of G_0 using colors $\{1, 2, 3, 4\}$ [6, 9]. If for some $i, c(x_i) \neq c(y_i)$, we recolor each vertex in $N^2(y_i)$ that is colored $c(y_i)$ with color 5 and then at y_i we perform a $(c(x_i), c(y_i))$-Kempe change so that y_i gets the same color as x_i. Since $\text{dist}(x_i, y_i) \geq 6$, this recoloring is valid. Next we transfer this coloring back to G_1. For each $i \leq s$ for which C''_i has even length, we 2-color C''_i with $\{c(x_i), 5\}$, and for each C''_i with a vertex v_i of degree 4, we 2-color C''_i with $\{c(x_i), 5\}$ except for v_i which is colored last with whatever color is available.

Next we correct the coloring for the vertices in $P \cap G_1$ that were not colored correctly by c. Since $\text{dist}(P, C''_i) \geq 5, c(p)$ is one of $\{1, 2, 3, 4\}$. If any $p \in P$ was precolored with color 5, its color can be changed immediately. If p was precolored with, say 1, and $c(p) = 2$, then we recolor all neighbors of p that are colored 1 with color 5, and then give p its desired color 1. Due to distance constraints, this produces a proper coloring on G_1; call this coloring c_1.

Finally, we add back and color the vertices deleted within contractible 3- and 4-cycles lying within $G_{C'_i,4}$ for each $i \leq s$. Let C be a contractible 3-cycle in $G_{C'_i,4}$ with non-empty interior in G. Thus $\text{dist}(C, C'_i) \leq 4$. Call the plane graph induced by C and its interior H. H may contain some elements of P, necessarily at distance at least 9 from C'_i and so at distance at least 5 from C. We begin with a 4-coloring of H that agrees with the coloring of C from G_1. Then we correct the coloring of any $p \in P \cap H$ as in the preceding paragraph. Since $\text{dist}(P, C) > 2$, the coloring on C is not changed and the two colorings are consistent.

Suppose $C = u, v, x, y$ is a contractible 4-cycle within $G_{C'_i,4}$ that within G contains more than one interior vertex. Thus some interior vertices were deleted to form G_1. Let H be the plane graph on C and its interior in G. We may assume that in the coloring of $G_1, c_1(u) = 1$ and $c_1(v) = 2$.

Case (i). Suppose $c_1(x) = 1$. We transform the plane graph H by identifying the vertices u and x within the outer face of H. If $c_1(y) = 2$ (resp., 3), then we 4-color H using colors $\{1, 3, 4, 5\}$ (resp., $\{1, 2, 4, 5\}$) making sure that u and x (resp., u, v, and x) are correctly colored. Then we correct the coloring of any vertices $p \in H$ as above. Since again $\text{dist}(P, C) > 2$, the color 2 (resp., 3) is not placed on or adjacent to vertices of C. Then we assign color 2 to both v and y (resp., color 3 to y) for a coloring of H that is correct on P and agrees on the boundary with that of C in G_1.

Case (ii). Suppose $c_1(x) = 3$. We transform the plane graph H by adding an edge joining u and x in the outer face of H. If $c_1(y) = 2$ (resp., 4), we 4-color H with $\{1, 3, 4, 5\}$ (resp., $\{1, 2, 3, 5\}$) making sure u and x (resp., u, v, and x) are correctly colored. Then as in the preceding paragraph we correct the coloring on P and assign color 2 to v and y (resp., color 4 to y). In this way the 5-precoloring of P always extends to G. □

The corollary below is a specific version of Theorem 4. It is also a strengthening of the original result in the sense that the lower bound on the width is much smaller.

Corollary 7.1. *If G is embedded in a surface S of Euler genus $g^* > 0$ and $w(G) \geq 64(2^{g^*} - 1)$, then $\chi(G) \leq 5$.*

Proof. Follow the proof of Theorem 7 assuming that $P = \emptyset$. □

Corollary 7.2. *Suppose G is embedded in S, a surface of Euler genus $g^* > 0$ such that $w(G) \geq 64(2^{g^*} - 1)$. If $d(P) \geq 4$, then any $(5 + q)$-coloring of $G[P]$ in which each component is q-colored extends to a $(5 + q)$-coloring of G.*

Proof. Corollary 7.1 implies that G is 5-colorable and the result follows from Theorem 2. □

4. When a subgraph is precolored

We close with a 5-coloring extension theorem in which each precolored component is 2-colored. Theorem 8 is a specific realization of Theorem 6.

Theorem 8. *Suppose G is embedded in S a surface of Euler genus $g^* > 0$. If, for $1 \leq i \leq k$ diameter $(G[P_i]) \leq D$, $d(P) \geq 18$, and $w_P(G) \geq (16D + 408)(2^{g^*} - 1)$, then any 5-coloring of $G[P]$ in which each component is 2-colored extends to a 5-coloring of G.*

Proof. We omit details that are identical to those in the proof of Theorem 7. We assume that G is a triangulation of S. We know that G contains a planarizing collection of chordless cycles $C_1, C_2, \ldots, C_s$ where $s \leq g^*$ such that for $1 \leq i \leq j \leq s$, $\text{dist}(C_i, C_j) \geq 2d \geq 2D + 50$ [13, 15]. We let $C'_1, C'_2, \ldots, C'_s$ denote the cycles that detour around the components of $G[P]$. Specifically C'_j is homotopic to C_j, $\text{dist}(C'_j, P) \geq 9$, and $\text{dist}(C'_i, C'_j) \geq 2D + 50 - 2(D + 18) \geq 14$.

We create the graph G_1 by making the subgraph $G_{C'_i,4}$ orderly for each $i \leq s$. Then for $1 \leq i \leq s$ we apply Lemma 1 with $C'_i = C_1$. We let C''_i denote the resulting nice cycle, homotopic to C'_i and lying within $G_{C'_i,4}$. This gives the final planarizing set of cycles $\{C''_1, \ldots, C''_s\}$. We have $\text{dist}(C''_i, C''_j) \geq 2d - 36 - 8 = 6$ and $\text{dist}(P, C''_i) \geq 5$.

Next we form a plane triangulation in two stages. First we cut out the planarizing cycles and replace each with either one or two vertices. Their distance from P is at least 5 and their distance from each other is at least 6. Next we look at P. Since $w_P(G) > 0$, each component of P is contractible. For $1 \leq i \leq k$ remove edges from $G[P_i]$ until the resulting component is a tree. Contract the ith component to obtain the vertex v_i and delete multiple edges. The resulting graph G_0 is a triangulation of the plane.

Suppose c is a 4-coloring of G_0 using colors $\{1, 2, 3, 4\}$. As before we arrange that for $1 \le j \le s$, if C_i'' is a 2-sided cycle, then $c(x_i) = c(y_i)$. We then color the vertices in $G_1 - G[P]$ by transferring the coloring of G_0 and coloring each of the nice planarizing cycles. We still need to fix the coloring in $G[P]$. For $1 \le i \le k$ we 2-color the vertices of $G[P_i]$ using the colors $c(v_i)$ and 5. The color classes in this 2-coloring are identical to the color classes in the hypothesized precoloring of $G[P_i]$.

Suppose for one particular $i, c(v_i) = 1$ and that the two color classes in $G[P_i]$ are assigned the colors 2 and 3 in the precoloring. First perform a (3, 5)-Kempe change at every vertex in $G[P_i]$ that is colored 5. Now every vertex in $G[P_i]$ that is supposed to be colored 3 is. Second perform a (1, 5)-Kempe change at every vertex in $G[P_i]$ that is colored 1. Finally, perform a (2, 5)-Kempe change at every vertex in $G[P_i]$ that is currently colored 5. These three Kempe changes have the effect of making the colors on $G[P_i]$ agree with those of the precoloring. Furthermore these color changes are confined to $P_i \cup N(P_i) \cup N^2(P_i) \cup N^3(P_i)$. The arguments for other possible precolorings are either similar or simpler [3].

The last step is to color the vertices of G that were deleted to create G_1. All of these vertices are interior to contractible 3- or 4-cycles that are within distance 4 of some C_i'. We will color these independently in exactly the same manner as was done in the proof of Theorem 7. $\square$

If one wanted an extension theorem in which a 6-coloring of $G[P]$ with each component 2-colored extends to a 6-coloring of all of G, then $d(P) \ge 18$ is needed. The required hypothesis on the relative width is that $w_P(G) \ge 208(2^{g^*}-1)$. This latter hypothesis can be replaced by assuming both that diameter $(G[P_i]) \le D$ and $w(G) \ge \frac{18}{17}(D+1)208(2^{g^*}-1)$. In either case the proofs are easier than the proof of Theorem 8.

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