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SUMSETS CONTAINING POWERS OF AN INTEGER

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Let $n \geq 1$, and let m be an integer with $m \geq 2$. We show that if a subset A of the interval $[0,n]$ satisfies that $0 \in A$ and $|A| > 1+n/2$, then mA, the set of the sum of m (not necessarily distinct) elements in A, has a power of m. This result is best possible in the case that m is odd.

1. Introduction

First of all we prepare some notations. Let $h \geq 2$, and let $A_1, A_2, \dots, A_h$ be sets of integers. We denote by $A_1 + A_2 + \cdots + A_h$ the set of all integers of the form $a_1 + a_2 + \cdots + a_h$, where $a_i \in A_i$ for $i = 1, 2, \dots, h$. We call it the *sumset*. If every A_i is the same, say A, then we denote $A_1 + A_2 + \cdots + A_h$ by hA. Let $n \geq 1$. A set A of nonnegative integers is called *normal* if it contains 0 and the greatest common divisor of all elements of A is 1. For each integer d, we define the set $d * A := \{da : a \in A\}$.

Erdős and Freud had conjectured that if $A \subseteq [1,n]$ with $|A| > n/3$, then some power of 2 can be written as the sum of distinct elements of A. This was proved by Erdős and Freiman [1], and sharpened by Nathanson and Sárkőzy [5]. Recently Lev [2] proved the following:

Theorem 1.1. *Let $n \geq 1$, and let A be a set of integers contained in the interval $[1,n]$. If $|A| > n/3$ then there is a power of 2 that can written as the sum of at most four (not necessarily distinct) elements of A.*

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In other word, if $A \subseteq [0,n]$ satisfies that $0 \in A$ and $|A| > 1 + n/3$ then $4A$ has a power of 2. This result is best possible in the sense: there is a subset A of $[0,n]$ such that $0 \in A$, $|A| = 1 + \lfloor n/3 \rfloor$ and the powers of 2 cannot be represented as sums of elements of A (indeed, A can be chosen as the multiples of 3). Moreover there is a subset A of $[0,n]$ such that $0 \in A$, $|A| > 1 + n/3$, and $3A$ has no power of 2.

Our main result in this paper is the following:

Theorem 1.2. *Let m be an integer with $m \geq 2$. Then mA contains a power of m whenever $A \subseteq [0,n]$ satisfies that $0 \in A$ and $|A| > 1 + n/2$.*

Theorem 1.2 is best possible in the case that m is odd. Let A be the set of all integers contained in the interval $[0,n]$ that are divisible by 2. Then $|A| = 1 + \lfloor n/2 \rfloor$. Since every sum of elements of A is divisible by 2, no power of m can be written as the sum of elements of A. Moreover we can construct examples of $A \subseteq [0,n]$ such that $0 \in A$ and $|A| > 1 + n/2$, but $(m-1)A$ contains no power of m. Let $n = m^r + 2$ where $r \geq 2$, and let

$$A = 2 * \left[0, \frac{n-1}{2}\right] \cup \{n\}.$$

Clearly $|A| = 1 + n/2 + 1/2$. Since $m-1$ is even, the set of odd integers of $(m-1)A$ is

$$\bigcup_{i=0}^{\frac{m-3}{2}} \left((m-2-2i) \left(2 * \left[0, \frac{n-1}{2}\right] \right) + \{(2i+1)n\} \right),$$

which is contained in the interval $[n, (m-1)n-1]$. Since $(m-1)n - 1 \leq m^{r+1} - (m(m-2)+3) < m^{r+1}$, $(m-1)A$ has no power of m.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. The next lemma will be used to prove the theorem.

Lemma 2.1. *Let $n \geq 1$, and let m be an integer with $m \geq 2$. If $A \subseteq [0,n]$ and $B \subseteq [0,(m-1)n]$ are such that $0 \in A$, $0 \in B$, and*

$$|A| + |B| > (m-1)n + 1$$

then the sumset $A + B$ contains a power of m.

Proof. Suppose on the contrary that the sumset $A+B$ contains no power of m. We may assume that $n \geq 2$. By induction on n, we have that $|A \cap [0, n-1]| + |B \cap [0, (m-1)(n-1)]| \leq (m-1)(n-1)+1$. Therefore $n \in A$ and $b := (m-1)n \in B$. Let r be an integer such that $m^r < b < m^{r+1}$. We consider two cases.

Case 1. $n < m^r$. Let $A_1 = (-1)*A + \{m^r\}$. Then $A_1 \subseteq [0, b]$, and moreover A_1 and B have no common element. Hence $|A| + |B| = |A_1| + |B| \leq (m-1)n+1$. This is a contradiction.

Case 2. $n > m^r$. Note that $b+n > m^{r+1}$. Let $A_1 = [0, b] \cap ((-1)*A + \{m^{r+1}\})$ and let $B_1 = B \cap [m^{r+1}-n, b]$. Clearly $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 \subseteq [m^{r+1}-n, b]$. If $m^{r+1} - b > 1$, then by induction on n, we have that

$$|A \cap [0, m^{r+1}-b-1]| + |B \cap [0, (m-1)(m^{r+1}-b-1)]| \leq (m-1)(m^{r+1}-b-1)+1,$$

and so

$$\begin{aligned} |A| + |B| &\leq (m-1)(m^{r+1}-b-1) + 1 + |A_1| + |B_1| \\ &\quad + |[(m-1)(m^{r+1}-b-1)+1, m^{r+1}-n-1]| \\ &\leq (m-1)n + 1. \end{aligned}$$

Otherwise, $m^{r+1} = b+1$. If $m = 2$, then $|A_1| + |B_1| \leq n-1$ because neither A_1 nor B_1 contains 2^r. If $m \geq 3$, then $|B \cap [0, m^{r+1}-n-1]| \leq m^{r+1}-n-1$ because $1 \notin B$. Therefore we have that

$$\begin{aligned} |A| + |B| &\leq 1 + |B \cap [0, m^{r+1}-n-1]| + |A_1| + |B_1| \\ &\leq (m-1)n + 1, \end{aligned}$$

which is a contradiction. ∎

Next we use Lev's theorem [3]. This is a very important result which is a generalization of Freiman's theorem (see [4]). Let $n+1 \geq k \geq 1$. By l we denote the positive integer satisfying

$$l(k-2) \leq n-1 \leq (l+1)(k-2). \tag{1}$$

Theorem 2.2. *If A is a normal k-element subset of the interval $[0, n]$, then for $h \geq 1$*

$$|hA| \geq \begin{cases} \dfrac{h(h+1)(k-2)}{2} + h + 1 & \text{if } h \leq l; \\[2ex] \dfrac{l(l+1)(k-2)}{2} + l + 1 + (h-l)n & \text{if } h \geq l. \end{cases}$$

Proof of Theorem 1.2. Let A be a subset of the interval $[0,n]$ such that $0\in A$ and $|A|>1+n/2$. We may assume that $n\in A$. Then A is normal. Since $2(|A|-2)>n-2$, we can see that the inequality (1) holds with $k=|A|$ and $l=1$. By Theorem 2.2,

$$|(m-1)A| \geq (|A|-2)+2+(m-2)n > (m-3/2)n+1,$$

and so

$$|A|+|(m-1)A| > (m-1)n+1.$$

From Lemma 2.1 we have that $mA=A+(m-1)A$ contains a power of m. ∎

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