Good covers are algorithmically unrecognizable

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Abstract

A *good cover* in $\mathbb{R}^d$ is a collection of open contractible sets in $\mathbb{R}^d$ such that the intersection of any subcollection is either contractible or empty. Motivated by an analogy with convex sets, intersection patterns of good covers were studied intensively. Our main result is that intersection patterns of good covers are algorithmically unrecognizable.

More precisely, the intersection pattern of a good cover can be stored in a simplicial complex called *nerve* which records which subfamilies of the good cover intersect. A simplicial complex is *topologically d-representable* if it is isomorphic to the nerve of a good cover in $\mathbb{R}^d$. We prove that it is algorithmically undecidable whether a given simplicial complex is topologically d-representable for any fixed $d \geq 5$.

As an auxiliary result we prove that if a simplicial complex is PL embeddable into $\mathbb{R}^d$, then it is topologically d-

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representable. We also supply this result with showing that if a "sufficiently fine" subdivision of a k-dimensional complex is d-representable and $k \leq \frac{2d-3}{3}$, then the complex is PL embeddable into $\mathbb{R}^d$.

1 Introduction

Many results in discrete geometry are devoted to studying intersection patterns of convex sets. A pioneering result in this respect is the Helly theorem [Hel23]. It states that whenever $C_1, \ldots, C_n$ are convex sets in $\mathbb{R}^d$, $n \geq d+1$, such that the intersection of any $d+1$ of these sets is nonempty, then the intersection of all sets is nonempty. Many results of similar flavor are known and the interested reader is referred to the survey paper [Tan11b] for more details.

Nerves and d-representable complexes. For a collection of sets, its intersection pattern can be encoded into a combinatorial object that is called the nerve of the collection.

Consider a collection of sets $\mathcal{F} = \{F_1, \ldots, F_n\}$. The *nerve* of $\mathcal{F}$ is the simplicial complex[1] whose k-dimensional faces are the subcollections $\{F_{i_1}, \ldots, F_{i_k}\}$ of $\mathcal{F}$ such that $F_{i_1} \cap \ldots \cap F_{i_k} \neq \emptyset$. In particular, the nerve of $\mathcal{F}$ has n vertices $F_1, \ldots, F_n$ (provided that F_i are nonempty).

Definition 1.1. A *convex cover* in $\mathbb{R}^d$ is a finite collection of open convex sets $U_1, \ldots, U_n \subset \mathbb{R}^d$.

Remark 1.2. Note that we do not require $\bigcup_{i=1}^n U_i = \mathbb{R}^d$. The word 'cover' should not be misleading.

Definition 1.3. A simplicial complex is *d-representable* if it is isomorphic to the nerve of a convex cover in $\mathbb{R}^d$.

Topological d-representability. The following generalization of a convex cover is rather well-known in topology.

[1] We briefly recall simplicial complexes and related definitions in section 2.

Definition 1.4. A *good cover* in $\mathbb{R}^d$ is a finite collection of open sets $U_1, \ldots, U_n$ in $\mathbb{R}^d$ such that the intersection $U_{i_1} \cap \ldots \cap U_{i_k}$ of any subcollection $\{U_{i_1}, \ldots, U_{i_k}\}$ is either empty or contractible. (In particular, U_i are contractible.)

Remark 1.5. A convex cover is a good cover.

Definition 1.6. A simplicial complex is *topologically d-representable*[2] if it is isomorphic to the nerve of a good cover in $\mathbb{R}^d$.

Classifying intersection patterns of convex (or good) covers in $\mathbb{R}^d$ is equivalent to classifying d-representable (resp. topologically d-representable) complexes.

Intersection patterns (formally, nerves) of good covers inherit many properties of intersection patterns of convex covers. For example, the Helly theorem was generalized to good covers again by Helly [Hel30]. Another example is the well-known Nerve theorem, see Theorem 2.1 below. (Probably, this theorem is the main reason that makes good covers easier to study than arbitrary collections of non-convex sets in $\mathbb{R}^d$.) Other examples include topological versions of various Helly-type theorems [AKMM02, KM05, KM08].

The main result of our paper, Theorem 1.8, is in the opposite spirit. We show that from the general algorithmic viewpoint, intersection patterns of good covers behave differently (in fact, much worse) than intersection patterns of convex covers.

First, recall the following theorem.

Theorem 1.7 ([Weg67], see also [Tan11b])**.** *It is algorithmically* decidable *whether a given simplicial complex is d-representable. (There is actually a PSPACE algorithm for recognition d-representable simplicial complexes.)*

As we will now show, the situation with topological d-representability is completely different. The main goal of our paper is to prove the following result.

[2] Topological d-representability was first introduced in [Tan12]. However, the definition was slightly different; see Remark 1.9.

Theorem 1.8 (main result)**.** *It is algorithmically* undecidable *whether a given simplicial complex is topologically d-representable.*

Algorithmically undecidable problems. A decision problem is the following question: given a finite input string s (over a finite alphabet), decide whether s satisfies a certain property P. Roughly speaking, a decision computational problem is algorithmically undecidable if there is no algorithm that would solve this problem for every string s. More precisely, there is no Turing machine solving this problem. However, the reader is not assumed to have background in Turing machines, since the details about Turing machines are hidden in reduction of our problem to famous Novikov's theorem (see Theorem 3.1 below).

Undecidable problems naturally appear in algebra. For instance the word problem for groups or semigroups is known to be undecidable [Pos47, Nov55]. These problems also reflect in topology. For example it is algorithmically undecidable whether the fundamental group of a given complex is trivial since the word problem reduces to triviality of the fundamental group. Another example is the abovementioned Novikov's theorem. Briefly, it states that for each $d \geq 5$ it is algorithmically undecidable whether a given simplicial complex is homeomorphic to the d-sphere.

In combinatorial geometry, undecidable problems are not so frequent (in the authors' opinion; depending on how broadly combinatorial geometry is considered). We should mention Wang's tiling problem proved undecidable by Berger [Ber66] as an example. Our problem is actually on the borderline area between topology and combinatorial geometry. We hope that our approach could have consequences in other problems in combinatorial geometry.

Remark 1.9. Let us call a simplicial complex *topologically d-representable by balls* if it is a nerve of a good cover $\{U_i\}$ in $\mathbb{R}^d$ such that any intersection $\{U_{i_1}, \ldots, U_{i_k}\}$, unless it is empty, is not only contractible, but even homeomorphic to the open d-ball.

In [Tan12], topological d-representability by balls was introduced as topological d-representability (in order to get a stronger result with this definition). Definition 1.6 of topological d-representability that we use in this paper is probably more standard in the literature, see, e.g., [AKMM02, KM05]. (These papers do not define topological d-representability; however, they actually prove some properties of topologically d-representable complexes.) For completeness we also prove the following modification of Theorem 1.8.

Theorem 1.10. *It is algorithmically undecidable whether a given simplicial complex is topologically d-representable by balls.*

Formally, neither does Theorem 1.8 imply Theorem 1.10 nor vice versa. But we prove them simultaneously in this paper.

2 Preliminaries

In this section, we quickly recall some basic definitions and notations mostly concerning simplicial complexes. A reader new to the topic might also want to see more substantial literature [Mat03, Hat01, Mun84, RS72]. We recommend to consult preliminaries only if the need arises.

Integers. For an integer n the symbol $[n]$ denotes the set $\{1, 2, \dots, n\}$.

Abstract simplicial complexes. Let V be a finite set. A collection K of subsets of V is a simplicial complex if, together with each $\alpha \in K$, we have $\beta \in K$ for every $\beta \subset \alpha$. Any $\sigma \in K$ with $\#\sigma = k + 1$ is called a k-dimensional simplex (or *face*) of K (by $\#\sigma$ we mean the number of elements of σ). The set V is the set of *vertices* of K. Usually we denote it by $V(K)$.

Let $U \subset V$. The induced subcomplex of K on U is given by $K[U] := \{\sigma \in K : \sigma \subseteq U\}$.

Let K, L be two simplicial complexes. A map $f : V(K) \to V(L)$ is a *simplicial map* if $f(\alpha) \in L$ for every $\alpha \in K$. Two complexes

K, L are said to be *isomorphic* if there is a bijective simplicial map $V(K) \to V(L)$.

Geometric realizations. We work a priori with abstract simplicial complexes. However, sometimes it is more convenient to work with geometrical realizations of abstract simplicial complexes. Given an abstract simplicial complex K, we chose a map $f\colon V(K) \to \mathbb{R}^m$ for sufficiently large m. Assume that f satisfy the following properties:

- The set $f(\alpha)$ is affinely independent for every $\alpha \in K$; and
- the convex hulls satisfy the relation $\operatorname{conv}(f(\alpha)) \cap \operatorname{conv}(f(\beta)) = \operatorname{conv}(f(\alpha \cap \beta))$ for every $\alpha, \beta \in K$.

If m is large enough, then such an f exists. For example, a map sending vertices of K injectively to the vertices of a (geometric) simplex in $\mathbb{R}^m$ is a suitable choice.

For a face $\alpha \in K$ we have the *geometric realization* of this face

$$|\alpha| := \operatorname{conv}\{f(v) \colon v \in \alpha\}.$$

We also have the *geometric realization* of any subcomplex $X \subset K$ given by

$$|X| := \bigcup_{\alpha \in X} |\alpha|.$$

Every complex K has a geometric realization $|K|$ and any two geometric realizations of a given complex are homeomorphic. We will assume that every complex has a fixed geometric realization although, in some cases, we keep the right to determine the particular choice.

If there is no risk of confusing the reader, we write X instead of $|X|$ for a subcomplex X of K. For example, if we say that complexes K and L are homeomorphic, we actually mean that $|K|$ and $|L|$ are homeomorphic.

Subdivisions. Let K, K' be simplicial complexes. We say that K' is a subdivision of K if $|K| = |K'|$ and for each face $\sigma' \in K'$ there is $\sigma \in K$ such that $|\sigma'| \subseteq |\sigma|$. Note that this definition a priori depends on the choice of the geometric realizations. However, this is not a problem for us if we fix a realization for every complex as we mentioned above.

PL maps and embeddings. Let K, L be two simplicial complexes. A continuous map $|K| \to |L|$ is called *PL (piecewise-linear)* if it is linear on the simplices of a subdivision K' of K. Then, by [RS72, 2.14], there is a subdivision L' of L such that f maps any simplex of K' to a simplex of L' and thus induces a simplicial map $V(K') \to V(L')$.

A PL map which is a homeomorphism is called a *PL homeomorphism.*

A *PL d-ball* is a simplicial complex PL homeomorphic to the d-simplex Δ^d. A *PL d-sphere* is a simplicial complex PL homeomorphic to the boundary of the d-simplex $\partial\Delta^d$. We remark that for d large enough there are known examples of simplicial complexes homeomorphic to the d-ball (resp. d-sphere) but which are not a PL d-ball (resp. PL d-sphere).

A *PL embedding* of a simplicial complex K into $\mathbb{R}^d$ is an injective map $|K| \to \mathbb{R}^d$ that is linear on the faces of K' where K' is some subdivision of K. A PL d-ball always PL embeds into $\mathbb{R}^d$ since the d-simplex PL embeds into $\mathbb{R}^d$. When we remove a simplex of maximum dimension from a PL d-sphere we obtain a PL d-ball [RS72, Corollary 3.13].[3]

The Nerve Theorem. We need the following version of the Nerve Theorem. The homotopy version is usually attributed to Borsuk [Bor48]. We use the formulation from Hatcher's book [Hat01].

Theorem 2.1 ([Hat01, 4G.3])**.** *If $\mathcal{U}$ is a collection of open sets in a paracompact space X such that $\bigcup \mathcal{U} = X$ and every nonempty intersection of finitely many sets in $\mathcal{U}$ is contractible, then X is homotopy equivalent to the nerve of $\mathcal{U}$.*

[3]Note that balls and spheres in the statement of Corollary 3.13 in [RS72] are a priori assumed PL.

For further use we recall that any subset of $\mathbb{R}^d$ or S^d is a paracompact space.

Homology balls and homology spheres. A *homology d-sphere* is a (topological) d-manifold with the same singular homology as the d-sphere. Similarly, a *homology d-ball* is a d-manifold with boundary which has the same singular homology as the d-ball.

Alexander duality and Čech cohomology. As a supplementary tool we also need Alexander duality. Roughly speaking, Alexander duality relates the cohomology of a "nice" closed subset K of S^d with the homology of $S^d \setminus K$. If we do not know whether K is "nice" (which will be our case), then the ordinary cohomology must be replaced with *Čech cohomology.* In order to define Čech cohomology, we would need too many preliminaries. Therefore we rather prefer to use it as a "black box" while referring to the literature for statements we need.

Here is a version of Alexander duality we need [Pra07, Theorem 5.7]:

Theorem 2.2 (Alexander duality)**.** *If $A \subsetneq S^d$ is a closed set, then*

$$\check{\tilde{H}}^k(A) \cong \tilde{H}_{d-k-1}(S^d \setminus A)$$

for $0 \leq k \leq n-1$. Here $\check{\tilde{H}}^*$ stands for reduced Čech cohomology and $\tilde{H}_*$ stands for reduced singular homology.

Lemma 2.3 ([ES52, exercise 3, p. 254])**.** *Let $X \subseteq S^d$. Then the (non-reduced) Čech cohomology group $\check{H}^0(X)$ is isomorphic to the group of continuous functions $X \to \mathbb{Z}$ where $\mathbb{Z}$ is equipped with discrete topology.*

For clarity, the following lemma summarizes all consequences of Alexander duality we will need.

Lemma 2.4. *Let M and N be two open subsets of S^d, $d \geq 2$. If M is homotopy equivalent to S^{d-1} and $H_{d-1}(N) = 0$, then* *(a)* $S^d \setminus M$ *contains exactly two components;*

(b) $S^d \setminus N$ *is connected; and*

(c) $H_{d-1}(M \cup C) = 0$ *where* C *is any of the components of* $S^d \setminus M$.

Proof. We prove the part (a) first. Let $A := S^d \setminus M$. Then $\tilde{\check{H}}^0(A) \cong \mathbb{Z}$ by Alexander duality (Theorem 2.2), and therefore $\check{H}^0(A) \cong \mathbb{Z} \oplus \tilde{\check{H}}^0(A) \cong \mathbb{Z}^2$.

If A contained three or more components then it could be partitioned into three disjoint clopen (closed and open) sets A_1, A_2 and A_3 (disconnected A can be partitioned into two clopen sets and then at least one of these sets can be partitioned again). Functions $A \to \mathbb{Z}$ constant on each of these clopen sets would be continuous. Therefore $\mathbb{Z}^3$ would be a subgroup of $\check{H}^0(A)$ due to Lemma 2.3. This contradicts $\check{H}^0(A) \cong \mathbb{Z}^2$.

Similarly, if A were connected, then $\check{H}^0(A) \cong \mathbb{Z}$, since every continuous function $A \to \mathbb{Z}$ would be constant. This contradicts $\check{H}^0(A) \cong \mathbb{Z}^2$ again.

Therefore part (a) is proved. Part (b) is analogous to (a), using $\tilde{\check{H}}^0(S^d \setminus N) \cong \mathbb{Z}$ which follows from the Alexander duality (reduced and non-reduced homology groups coincide in dimension $d-1$, since $d \geq 2$). It remains to prove (c).

Let C' be the second component of $A = S^d \setminus M$. Note that both C and C' are closed in S^d since A is closed in S^d and the number of components of A is finite, namely two. Using Lemma 2.3 again, we derive $\check{H}^0(C') \cong \mathbb{Z}$, and therefore $\tilde{\check{H}}^0(C') = 0$. Part (c) now follows from the Alexander duality.

□

3 The proof method

In this section we describe our proof method. On a general level, we follow the approach by Matoušek, Tancer and Wagner [MTW11] showing that it is algorithmically undecidable whether a given $(d-1)$-dimensional simplicial complex embeds in $\mathbb{R}^d$ (for $d \geq 5$). Some details are, however, more difficult to resolve in our case.

Our main ingredient is Novikov's theorem (Theorem 3.1). Using it we construct a sequence of simplicial complexes $\{C_i\}_{i=1}^{\infty}$ such that each C_i

- is either PL homeomorphic to the d-ball
- or has nontrivial fundamental group,

and there is no algorithm deciding which of the two cases holds. The main task is to show that C_i is topologically d-representable in the first case (this is rather straightforward but a bit technical; see Theorem 3.3) but not in the second (this is not so obvious and the reader might be also interested in the used technique; see Proposition 3.4. It uses a special feature of the combinatorial structure of C_i; see *collaring* below.)

Now we describe our method in more details.

Novikov's theorem. Novikov proved that it is algorithmically undecidable whether a given (CW-)complex is homeomorphic to a d-sphere if $d \geq 5$. We need the following variation of his theorem.

Theorem 3.1 (**Novikov**). *Let $d \geq 5$ be a fixed integer. There is an effectively constructible sequence of simplicial complexes Σ_i, $i \in \mathbb{N}$, with the following properties:*

(1) *Each $|\Sigma_i|$ is a homology d-sphere (in particular a manifold).*

(2) *For each i, either Σ_i is a* PL *d-sphere, or the fundamental group of Σ_i is nontrivial (in particular, Σ_i is not homeomorphic to the d-sphere).*

(3) *There is no algorithm that decides for every given Σ_i which of the two cases holds.*

A proof of Theorem 3.1 follows from the exposition by Nabutovsky; see the appendix of [Nab95]. Indeed Nabutovsky constructs a sequence of polynomials such that it is algorithmically undecidable whether their zero set is homeomorphic to a d-sphere. These zero sets are always smooth manifolds, and if they are homeomorphic to a d-sphere, they are in addition diffeomorphic to the standard d-sphere. Such smooth manifolds have a natural PL-structure [Whi40] and their triangulations can be found algorithmically [BPR06, Remark 11.19] (see Remark 12.35 if you consult the first edition). We conclude by remarking that in case of triangulating standard (smooth) d-sphere we obtain a PL-sphere.

Our task is to transform this result into undecidability of recognition of topologically d-representable complexes (for $d \geq 5$).

Removing a simplex. Let B_i be the simplicial complex obtained from Σ_i by removing a d-simplex. Each B_i is a homology d-ball; B_i is embeddable into $\mathbb{R}^d$ if and only if Σ_i is a PL sphere (which is algorithmically unrecognizable).

A straightforward approach (motivated by [MTW11]) would be to prove the following conjecture.

Conjecture 3.2. *A simplicial complex K is PL embeddable into $\mathbb{R}^d$ if and only if its barycentric subdivision is topologically d-representable.*

An affirmative answer to this conjecture implies Theorem 1.8 (our main result) if it is used for the barycentric subdivisions of the sets B_i. (For brevity of this part, the definition of barycentric subdivision is postponed to section 4.)

We prove the ‘only if’ part even in a stronger form (Theorem 3.3) but we could not prove the ‘if’ part in such generality, or even for $K = B_i$. So we will modify the simplicial complexes B_i and obtain a new sequence of complexes C_i. Using some new combinatorial features of C_i we are able to prove that they are PL embeddable into $\mathbb{R}^d$ if and only if they are topologically d-representable.

In section 6 we also prove Conjecture 3.2 in case that $\dim K \leq \frac{2d-3}{2}$. This range is unfortunately not sufficient for our main result; however, we still hope that it is an interesting supplementary result.

Collaring. Fix Σ_i and B_i. Let $U := \{u_1, \ldots, u_{d+1}\}$ be the vertices of the simplex removed from Σ_i and $V = \{v_1, \ldots, v_{d+1}\}$ be additional

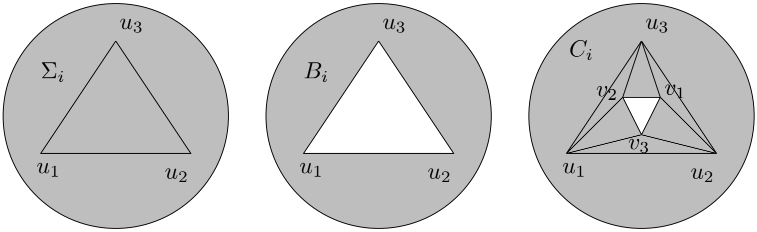

Figure 1: The complexes Σ_i, B_i and C_i

points (not the vertices of Σ_i).

We now create a simplicial complex Γ with vertices $u_1, \ldots, u_{d+1}$, $v_1, \ldots, v_{d+1}$. The set of simplices of Γ is the following:

$$\{\sigma \subset U \cup V : \sigma \neq U, V \text{ and } \{u_j, v_j\} \not\subseteq \sigma \text{ for any } j \in \{1, \ldots, d+1\}\}.$$

If we did not require $\sigma \neq U, V$ we would obtain a d-dimensional *crosspolytope* (see, e.g., [Mat03, p. 11]). Thus Γ is isomorphic to a d-dimensional crosspolytope minus two opposite d-simplices. In particular, Γ is homeomorphic to $S^{d-1} \times [0, 1]$.

Now set $C_i := B_i \cup \Gamma$, see Figure 1. Informally, we attached a cylinder ('collar') Γ to B_i and obtained C_i. Clearly, C_i is homeomorphic to B_i.

We are going to show that C_i is topologically d-representable if and only if Σ_i is homeomorphic to the d-sphere. We will split this task into two statements proved in separate sections.

Theorem 3.3. *Let K be a simplicial complex PL embeddable into $\mathbb{R}^d$. Then K is topologically d-representable by balls (see Remark 1.9).*

Proposition 3.4. *Let i be such that Σ_i has a nontrivial fundamental group. Then C_i is not topologically d-representable.*

Theorem 3.3 is proved in section 4; Proposition 3.4 is proved in section 5. The implication in Theorem 3.3 cannot be reverted, since a simplex of arbitrary high dimension is topologically d-representable by balls; however it is not PL embeddable into $\mathbb{R}^d$ if the dimension of the simplex exceeds d. Similarly, this example shows that an analogue of Conjecture 3.2 running as follows: a simplicial complex is d-representable if and only if it PL embeds into $\mathbb{R}^d$, is false.

We conclude this section by summarizing the above mentioned steps into a proof of Theorem 1.8 (and Theorem 1.10 as well).

Proof of Theorem 1.8 and Theorem 1.10. Let $\{C_i\}_{i=1}^{\infty}$ be the sequence of simplicial complexes constructed in this section.

If i is such that Σ_i is not homeomorphic to a d-sphere, then C_i is not topologically d-representable by Proposition 3.4. (And therefore C_i is neither topologically d-representable by balls.)

If i is such that Σ_i is homeomorphic to a d-sphere, then Σ_i is actually a PL d-sphere by Theorem 3.1. Let $\vartheta := \{v_1, \ldots, v_{d+1}\}$. Then $C_i \cup \{\vartheta\}$ can be regarded as a subdivision of Σ_i, and therefore $C_i \cup \{\vartheta\}$ is a PL d-sphere. Consequently, C_i is a PL d-ball [RS72, Corollary 3.13]. So C_i PL embeds into $\mathbb{R}^d$, and hence C_i is topologically d-representable by balls by Theorem 3.3 (in particular, it is topologically d-representable).

□

4 Embeddable complexes are topologically representable

In this section we prove Theorem 3.3.

Suppose that K is a simplicial complex and $f : |K| \to \mathbb{R}^d$ is a PL embedding. Let V be the set of vertices of K. We have to construct a topological d-representation of K, i.e., a family of sets $\{U_v\}_{v \in V}$, $U_v \subset \mathbb{R}^d$ such that

(a1) the nerve of $\{U_v\}$ is (isomorphic to) K; and

(a2) the sets U_v and all their intersections are either homeomorphic to an open d-ball or empty.

Plan of the proof. The proof contains two steps. First, we construct a family $\{X_v\}_{v \in V}$ of certain subcomplexes $X_v \subset K$ such that

(b1) the nerve of $\{|X_v|\}$ is K; and

(b2) the sets $|X_v|$ and all their intersections are either (simplicial) cones or empty.

Second, we consider the images $f(|X_v|) \subseteq \mathbb{R}^d$. The family $\{f(|X_v|)\}$ has property (a1), but not property (a2). We will introduce U_v as a properly defined open neighborhood of $f(|X_v|)$ in $\mathbb{R}^d$ and show that $\{U_v\}$ is a good d-representation by balls of K.

See Figure 2 while following the construction.

Subdivisions and stars. Let L be a simplicial complex. We will recall two notions that we will need further: the *barycentric subdivision* of L, and the *star* of a vertex $u \in V(L)$.

Formally, the *barycentric subdivision* $\operatorname{sd} K$ of a simplicial complex K is a simplicial complex whose vertices are the faces of K except the empty face; and the faces of $\operatorname{sd} K$ are the chains of faces of K

$$\Lambda = \{\sigma_1, \dots, \sigma_m\} \text{ such that } \emptyset \neq \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_m.$$

If there is no risk of confusing reader we simplify the notation by writing

$$\Lambda = \{\sigma_1 \subsetneq \cdots \subsetneq \sigma_m\}.$$

In the geometric setting, we can set $|K| = |\operatorname{sd} K|$ in such a way that a vertex of $\operatorname{sd} K$ corresponding to a simplex $\sigma \in K$ is situated in the barycentre of $|\sigma| \subset |K|$.

Let u be a vertex of L. The (closed) *star* of u in L is defined as $\operatorname{st}(u, L) := \{\sigma \in L \colon u \cup \sigma \in L\}$.

First step. A cover $\{|X_v|\}$ inside $|K|$. For each $v \in V$, denote $X_v := \operatorname{st}(v, \operatorname{sd} K)$. It is a subcomplex of $\operatorname{sd} K$. For $S \subseteq V$, denote $X_S := \bigcap_{v \in S} X_v$.

The following claim says that the geometric realizations $|X_v|$ of X_v form a cover with properties (b1) and (b2) announced in the plan of the proof.

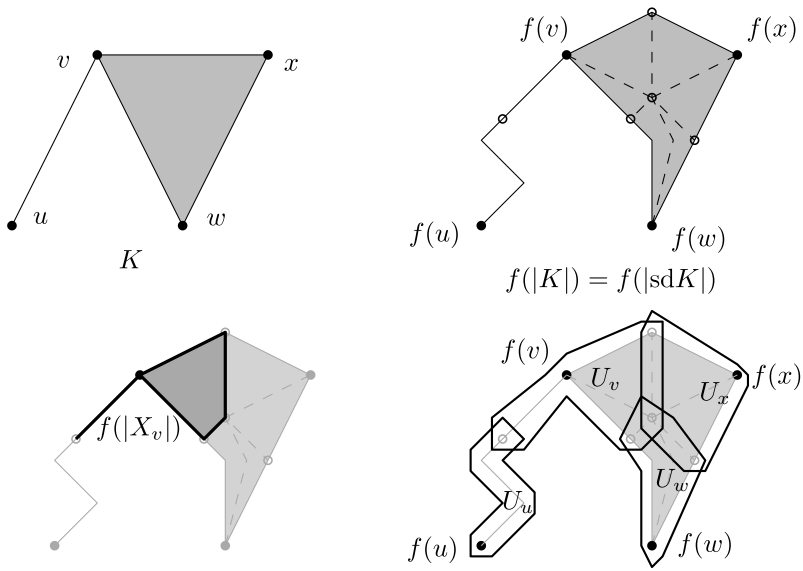

Figure 2: A complex K (top left); a PL embedding f of $|K|$ into $\mathbb{R}^2$ (top right); a set $f(|X_v|)$ – image of the star of v in the barycentric subdivision of K (bottom left); and the sets $\{U_v\}$ forming a topological d-representation of K (bottom right).

Claim 4.1. *For every $S \subseteq V$, X_S is nonempty if and only if $S \in K$. If X_S is nonempty, then it is a cone.*

Proof. According to the definitions of star and barycentric subdivision we have:

$$\mathrm{st}(\{v\}, \mathrm{sd}\, K) = \{\{\sigma_1 \subsetneq \cdots \subsetneq \sigma_m\} \in \mathrm{sd}\, K : v \in \sigma_i \text{ for every } i \in [m]\}.$$

Therefore

$$X_S = \bigcap_{v \in S} \mathrm{st}(\{v\}, \mathrm{sd}\, K) = \{\{\sigma_1 \subsetneq \cdots \subsetneq \sigma_m\} \in \mathrm{sd}\, K : S \subseteq \sigma_i \text{ for every } i \in [m]\}.$$

Hence X_S is nonempty if and only if $S \in K$. In addition, if $S \in K$, then $\bigcap_{v \in S} \operatorname{st}(\{v\}, \operatorname{sd} K)$ is a cone in $\operatorname{sd} K$ with apex $\{S\}$. □

Derived neighborhoods and collapsibility. Here we will briefly recall another concept of PL topology. Let $L \subset M$ be a simplicial embedding of a simplicial complex L into a simplicial d-manifold M. The *derived neighborhood* of L in M is the subcomplex $N(L)$ of $\operatorname{sd} \operatorname{sd} M$ whose geometric realization $|N(L)|$ is the union of all $|\sigma|$ such that $\sigma \in \operatorname{sd} \operatorname{sd} M$ is a d-simplex and $|\sigma| \cap |L| \neq \emptyset$. See Figure 3.

The definition of *collapsible* simplicial complexes is found in e.g. [RS72, p.39]. We will omit this definition since we use only some properties of collapsibility described in the following two lemmas.

Lemma 4.2. *[RS72, p.40] If a simplicial complex L is a cone over another simplicial complex, then $|L|$ is collapsible.*

Lemma 4.3. *[RS72, 3.27] Let $L \subset M$ be simplicial embedding of a simplicial complex L into a simplicial d-manifold M. If $|L|$ is collapsible, then $|N(L)|$ is PL homeomorphic to the d-ball.*

We also need Corollary 4.5 below which is implied by the following lemma. The lemma provides a combinatorial description of the derived neighborhood.

For a simplicial complex K we define a function $\mu\colon \operatorname{sd} K \to K$ that assigns to a chain $\Lambda \in \operatorname{sd} K$ the minimal element of this chain. That is, $\mu(\Lambda) = \sigma_1$ if

$$\Lambda = \{\sigma_1 \subsetneq \cdots \subsetneq \sigma_k\} \in \operatorname{sd} K.$$

Lemma 4.4. *Let $L \subset M$ be simplicial embedding of a simplicial complex L into a simplicial d-manifold M. Then*

$$N(L) = \{\sigma \in \operatorname{sd} \operatorname{sd} M \colon \mu(\mu(\sigma)) \in L\}.$$

Proof. Let $\sigma \in \operatorname{sd} \operatorname{sd} M$.

We first assume that $\mu(\mu(\sigma)) \in L$ and we will show that $\sigma \in N(L)$. From the definition of μ it follows that $\mu(\mu(\sigma)) \in \mu(\sigma)$; hence

$\{\mu(\mu(\sigma))\} \subseteq \mu(\sigma)$.[4] Consequently $h(\sigma) := \sigma \cup \{\{\mu(\mu(\sigma))\}\}$ is a simplex of $\operatorname{sd}\operatorname{sd} M$. (Note that it might happen that $h(\sigma) = \sigma$ if $\{\mu(\mu(\sigma))\} = \mu(\sigma)$.) The geometric realization $|h(\sigma)|$ intersects $|L|$ (in $|\{\mu(\mu(\sigma))\}|$ as a vertex of $\operatorname{sd} K$, that is, in the barycentre of the simplex $|\mu(\mu(\sigma))|$ of K). Therefore $\sigma \in N(L)$.

Before proving the second inclusion, we first realize that if $\vartheta \in M \setminus L$, then $|\operatorname{st}(\{\{\vartheta\}\}, \operatorname{sd}\operatorname{sd} M)|$ does not meet $|L|$. This is because $|\operatorname{st}(\{\{\vartheta\}\}, \operatorname{sd}\operatorname{sd} M)| \subseteq \operatorname{Int}|\operatorname{st}(\{\vartheta\}, \operatorname{sd} M)|$ (where Int denotes the interior), and $\operatorname{Int}|\operatorname{st}(\{\vartheta\}, \operatorname{sd} M)|$ does not meet $|L|$ since $\vartheta \notin L$.

Now we assume that $\mu(\mu(\sigma)) \notin L$. We will show that $\sigma \notin N(L)$. That is, we want to show that $|\tau| \cap |L| = \emptyset$ for every $\tau \in \operatorname{sd}\operatorname{sd} M$ with $\sigma \subseteq \tau$. See Figure 3. From the definition of μ the inclusion $\sigma \subseteq \tau$ implies $\mu(\tau) \subseteq \mu(\sigma)$. Applying once more, we get $\mu(\mu(\sigma)) \subseteq \mu(\mu(\tau))$. Therefore $\mu(\mu(\tau)) \notin L$ since $\mu(\mu(\sigma)) \notin L$. Similarly as before we have a simplex $h(\tau) := \tau \cup \{\{\mu(\mu(\tau))\}\}$ containing τ and therefore σ as well. However, $|h(\tau)| \cap |L| = \emptyset$ since $h(\tau) \in \operatorname{st}(\{\{\mu(\mu(\tau))\}\}, \operatorname{sd}\operatorname{sd} M)$, using the observation from the previous paragraph.

□

Corollary 4.5. *Let $L_1, L_2 \subset M$ be two simplicial embeddings of simplicial complexes L_1, L_2 into a simplicial d-manifold M. Then $N(L_1 \cap L_2) = N(L_1) \cap N(L_2)$.*

Proof. Let $\sigma \in \operatorname{sd}\operatorname{sd} M$. We have that $\sigma \in N(L_1) \cap N(L_2)$ if and only if $\mu(\mu(\sigma)) \in L_1$ and $\mu(\mu(\sigma) \in L_2$. This happens if and only if $\mu(\mu(\sigma)) \in L_1 \cap L_2$, that is, if and only if $\sigma \in N(L_1 \cap L_2)$. □

Second step. A good cover $\{U_v\}$ in $\mathbb{R}^d$. By [RS72, 2.14] there is a subdivision of $\operatorname{sd} K$ and a triangulation of $\mathbb{R}^d$ such that f maps any simplex to simplex in these triangulations. So f induces a simplicial map between these triangulations as abstract simplicial complexes. We denote this simplicial map by f again: further f will denote the

[4] Note that, purely formally, if v is a vertex of M, then $\{v\}$ is the corresponding vertex of $\operatorname{sd} M$, and $\{\{v\}\}$ the corresponding vertex of $\operatorname{sd}\operatorname{sd} M$. This explains the necessity of using iterated parentheses.

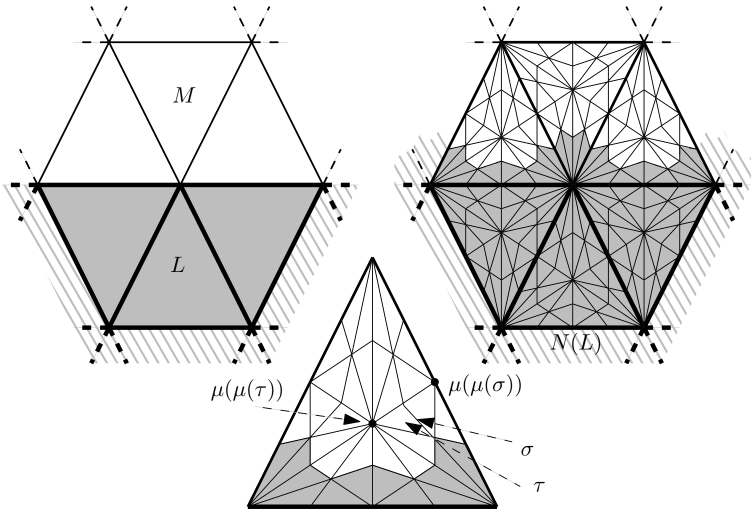

Figure 3: Derived neighborhoods: A complex L embedded in a triangulated manifold M (left). The derived neighborhood $N(L)$ (right). In addition one of the triangles is enlarged (bottom) with a particular choice of σ and τ such as in the proof of Lemma 4.4 (second inclusion). The notation is simplified; $\mu(\mu(\sigma))$ stands for $|\{\{\mu(\mu(\sigma))\}\}|$; σ stands for $|\sigma|$, etc.

simplicial map only. For $v \in V$, define $U_v := \mathrm{Int}\,|N(f(X_v))|$. Here the derived neighborhood is taken with respect to the triangulations above.

We conclude the section by proving Theorem 3.3.

Proof of Theorem 3.3. It suffices to prove that the sets U_v obtained above, and all their intersections, are (either empty or) d-balls. ¿From Claim 4.1 we know that $X_S = \bigcap_{v\in S} X_v$ is nonempty if and only if $S \in K$. If $S \in K$, by Claim 4.1 it is a cone, hence X_S is collapsible by Lemma 4.2, and so is $|\bigcap_{v\in S} f(X_v)|$. Consequently,

$|N(\bigcap_{v\in S} f(X_v))|$ is a d-ball by Lemma 4.3. Then by Corollary 4.5 $\bigcap_{v\in S} U_v = \mathrm{Int}\,|N(\bigcap_{v\in S} f(X_v))|$ is the interior of a d-ball, and thus an open d-ball. □

5 Nontrivial fundamental group is an obstruction

Proof of non-representability of C_i in non-ball case. Let us prove Proposition 3.4.

Let C_i be fixed. Suppose that there is a good cover in $\mathbb{R}^d$ whose nerve is isomorphic to C_i. Its elements consist of open subsets of $\mathbb{R}^d$ (further called *cells*), each cell corresponding to a vertex of C_i. Let S^d, the d-sphere, be the 1-point compactification of $\mathbb{R}^d \subset S^d$. Any subset in $\mathbb{R}^d$ will be automatically considered as a subset of S^d.

Recall that C_i has two special sets of vertices $U = \{u_i\}_{i=1}^{d+1}$, $V = \{v_i\}_{i=1}^{d+1}$ belonging to the 'collar' (see Figure 1). Let $\mathbf{U}_j$ (resp. $\mathbf{V}_j$) be the cell corresponding to the vertex u_j (resp. v_j) for $j = 1, \dots, d+1$. Denote $\mathbf{U} := \mathbf{U}_1 \cup \cdots \cup \mathbf{U}_{d+1}$, and $\mathbf{V} := \mathbf{V}_1 \cup \cdots \cup \mathbf{V}_{d+1}$. Moreover, let X be the set of vertices of C_i minus $U \cup V$ and let $\mathbf{X}$ be the union of the cells corresponding to vertices of X.

In our considerations we frequently use the Nerve Theorem (Theorem 2.1) without explicitly mentioning it (for instance $\mathbf{V}$ is homotopy equivalent to the subcomplex of C_i induced by vertices of V, which is homotopy equivalent to S^{d-1}, etc.).

By Alexander duality, more precisely by Lemma 2.4(a), $S^d \setminus \mathbf{V}$ has exactly two components. We know that $\mathbf{V}$ and $\mathbf{X}$ are disjoint since there is no edge connecting a vertex of X with a vertex of V (this is the place where we use the 'collar' structure of C_i). Thus we can denote by V_X the component of $S^d \setminus \mathbf{V}$ containing $\mathbf{X}$ and by V_Y the remaining one. See Figure 4.

Claim 5.1. *We have $V_X \subseteq \mathbf{X} \cup \mathbf{U}$.*

We first derive the result from the claim, and we prove the claim later.

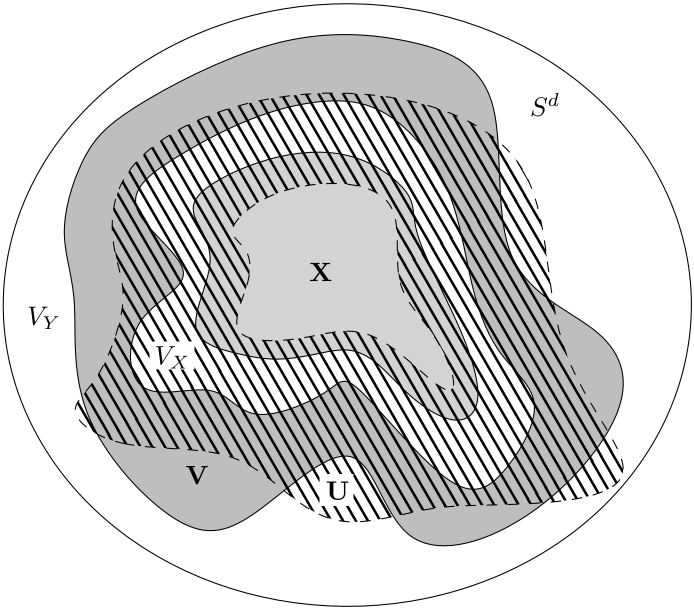

Figure 4: The sets **U**, **V**, **X**, V_X, and V_Y. The set **V** is grey, **U** is dashed, **X** is light grey, V_X is the component of $S^d \setminus \mathbf{V}$ inside **V** and V_Y is the other component.

Let us set $L := \mathbf{U} \cup \mathbf{X}$ and $M := \mathbf{U} \cup \mathbf{V} \cup V_Y$. We have $L \cup M = S^d$ by Claim 5.1. We also have $L \cap M = \mathbf{U}$, since **X** and V_Y are disjoint by the definition of V_Y, and we have already observed that **X** and **V** are disjoint. Both $L \cup M$ and $L \cap M$ have trivial fundamental group, thus L also has trivial fundamental group by Seifert–van Kampen theorem.

On the other hand L must have a nontrivial fundamental group, since it is homotopy equivalent to B_i by the Nerve Theorem. We obtain a contradiction as soon as we prove Claim 5.1.

In order to prove Claim 5.1 we need two other auxiliary claims. The first one does not seem new, but we could not find a reference for it.

Claim 5.2. *Let $A \subset B$ be two simplicial complexes which are represented by good covers. Let $\mathbf{A}$ (resp. $\mathbf{B}$) be the union of all cells in the representation of A (resp. B). Then the following diagram is commutative, in which the horizontal maps are inclusion-induced and the vertical maps are the isomorphisms induced by the homotopy equivalence from the Nerve Theorem.*

$$\begin{array}{ccc} H_k(A) & \longrightarrow & H_k(B) \\ \big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \cong} \\ H_k(\mathbf{A}) & \longrightarrow & H_k(\mathbf{B}) \end{array}$$

Claim 5.3. *We have $V_Y \not\subseteq \mathbf{U}$.*

Proof of Claim 5.2. For each cover (let it be the cover corresponding to A), there can be constructed [Hat01, 4G] a space $\Delta(A)$ together with projections $\mathrm{pr}_A : \Delta(A) \to A$ and $\mathrm{pr}_{\mathbf{A}} : \Delta(A) \to \mathbf{A}$ which are homotopy equivalences. (This is how the Nerve Theorem is generally proved.) Apply the same construction to B. It is easy to see from the definitions [Hat01, 4G] that we get $\Delta(A) \subset \Delta(B)$, $\mathrm{pr}_A = \mathrm{pr}_B|_{\Delta(A)}$ and $\mathrm{pr}_{\mathbf{A}} = \mathrm{pr}_{\mathbf{B}}|_{\Delta(A)}$. We thus obtain the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\subset} & B \\ {\scriptstyle \mathrm{pr}_A}\big\uparrow{\scriptstyle \sim} & & {\scriptstyle \mathrm{pr}_B}\big\uparrow{\scriptstyle \sim} \\ \Delta(A) & \longrightarrow & \Delta(B) \\ {\scriptstyle \mathrm{pr}_{\mathbf{A}}}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \mathrm{pr}_{\mathbf{B}}}\big\downarrow{\scriptstyle \sim} \\ \mathbf{A} & \xrightarrow{\subset} & \mathbf{B} \end{array}$$

The vertical maps are homotopy equivalences. Passing to homology, we obtain the commutative diagram

$$\begin{array}{ccc}
H_k(A) & \longrightarrow & H_k(B) \\
{\scriptstyle (\mathrm{pr}_A)_*}\big\uparrow{\scriptstyle \cong} & & {\scriptstyle (\mathrm{pr}_B)_*}\big\uparrow{\scriptstyle \cong} \\
H_k(\Delta(A)) & \longrightarrow & H_k(\Delta(B)) \\
{\scriptstyle (\mathrm{pr}_{\mathbf{A}})_*}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle (\mathrm{pr}_{\mathbf{B}})_*}\big\downarrow{\scriptstyle \cong} \\
H_k(\mathbf{A}) & \longrightarrow & H_k(\mathbf{B})
\end{array}$$

where the vertical maps are isomorphisms. □

Proof of Claim 5.3. Recall the subcomplex $\Gamma \subset C_i$ from the collaring procedure. Let $\Gamma[V]$ be the subcomplex of Γ generated by the set of vertices V. In this proof, we will abuse notation and write Γ, $\Gamma[V]$ instead of $|\Gamma|$, $|\Gamma[V]|$. We apply Claim 5.2 taking $A = \Gamma[V]$, $B = \Gamma$ and $k = d - 1$. We obtain the following commutative diagram.

$$\begin{array}{ccc}
H_{d-1}(\Gamma[V]) & \xrightarrow{\ f\ } & H_{d-1}(\Gamma) \\
\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \cong} \\
H_{d-1}(\mathbf{V}) & \xrightarrow{\ g\ } & H_{d-1}(\mathbf{U} \cup \mathbf{V})
\end{array}$$

As follows from the definition of Γ, the map f is the isomorphism $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$. On the other hand, if $V_Y \subset \mathbf{U}$, then g is the zero map. Indeed, under this assumption g is the composition of the inclusion-induced maps

$$H_{d-1}(\mathbf{V}) \to H_{d-1}(\mathbf{V} \cup V_Y) \to H_{d-1}(\mathbf{U} \cup \mathbf{V})$$

with $H_{d-1}(\mathbf{V} \cup V_Y) = 0$ due to Lemma 2.4(c), so $g = 0$. This contradicts to the commutativity of the diagram. □

Proof of Claim 5.1. The set $\mathbf{V} \cup \mathbf{U} \cup \mathbf{X}$ has trivial $(d-1)$st homology, since it is homotopy equivalent to C_i which is a homology ball. Hence $S^d \setminus (\mathbf{V} \cup \mathbf{U} \cup \mathbf{X})$ is connected due to Lemma 2.4(b). Thus $\mathbf{V} \cup \mathbf{U} \cup \mathbf{X}$

has to contain (exactly) one of the components of $S^d \setminus \mathbf{V}$. Hence $\mathbf{U} \cup \mathbf{X}$ has to contain V_X or V_Y. In addition $\mathbf{X}$ is disjoint with V_Y by the definition of V_X and $\mathbf{U}$ does not cover V_Y by Claim 5.3. The only remaining option is that $\mathbf{U} \cup \mathbf{X}$ covers V_X. $\square$

6 Topological d-representability in the metastable range

In this section we prove Conjecture 3.2 if $\dim K \leq \frac{2d-3}{3}$. More precisely, we prove the following result since the converse implication is already covered by Theorem 3.3:

Theorem 6.1. *Assume that K is a k-dimensional simplicial complex with $k \leq \frac{2d-3}{3}$. If $\operatorname{sd} K$, or any subdivision of $\operatorname{sd} K$, is topologically d-representable, then K PL embeds into $\mathbb{R}^d$.*

The assumption $k \leq \frac{2d-3}{3}$ is known as that the pair (k, d) belongs to the *metastable range* of a theorem of Haefliger and Weber. The contents of this section can be regarded as an extension of methods used in [Tan11a].

We need some preliminaries.

Haefliger-Weber Theorem. Let X be a compact topological space. The *deleted product* of a topological space X is the Cartesian product of X with itself minus the diagonal:

$$\tilde{X} := X \times X \setminus \{(x, x) \colon x \in X\}.$$

There is a natural $\mathbb{Z}_2$-action on $\tilde{X}$ given by swapping coordinates: $(x, y) \to (y, x)$. In sequel we assume that $\tilde{X}$ is equipped with this $\mathbb{Z}_2$-action. By S^{d-1}_- we also denote $(d-1)$-dimensional sphere equipped with the antipodal action $x \to -x$.

Let us assume that there exists an embedding $f \colon X \to \mathbb{R}^d$. The *Gauss map* $\tilde{f} \colon \tilde{X} \to S^{d-1}_-$ is the $\mathbb{Z}_2$-equivariant map given by formula

$$\tilde{f}(x, y) = \frac{f(x) - f(y)}{\|f(x) - f(y)\|}.$$

Therefore we know that the existence of embedding X into $\mathbb{R}^d$ implies the existence of $\mathbb{Z}_2$-equivariant map from $\tilde{X}$ to S_{-}^{d-1}.

The celebrated Haefliger-Weber Theorem ([Hae63, Web67]; see also [Sko08]) states that for polyhedra in the metastable range the existence of an embedding and the existence of the equivariant map are equivalent:

Theorem 6.2 (Haefliger-Weber). *Let X be a geometric realization of a k-dimensional simplicial complex. Let us also assume that $k \leq \frac{2d-3}{3}$. If there is a $\mathbb{Z}_2$-equivariant map $\tilde{X} \to S_{-}^{d-1}$, then X is PL embeddable into $\mathbb{R}^d$.*

Weakly injective maps and embeddings. Let K be a simplicial complex. We say that a map $f\colon |K| \to \mathbb{R}^d$ is *weakly injective* (with respect to K) if for every two disjoint simplices $\gamma, \delta \in K$ their images $f(|\gamma|)$ and $f(|\delta|)$ are disjoint as well.

Remark 6.3. Note that every injective map is weakly injective, but the converse is not true. In a weakly injective map the images of two faces sharing a vertex might intersect also in other points. Also the image of a single face might be self-intersecting or even degenerate.

For our purposes we need the following corollary of the Haefliger-Weber Theorem:

Corollary 6.4. *Let K be a k-dimensional simplicial complex and d be such that $k \leq \frac{2d-3}{3}$. Then the existence of a weakly injective map $f\colon |K| \to \mathbb{R}^d$ implies the existence of a PL embedding $|K| \to \mathbb{R}^d$.*

Proof. A simplicial deleted product of $|K|$ is a topological space consisting of products of pairs of disjoint simplices in $|K|$:

$$|\tilde{K}|_s := \{|\sigma| \times |\tau| \colon \sigma, \tau \in K; \sigma \cap \tau = \emptyset\}.$$

The existence of f implies that there is a $\mathbb{Z}_2$-equivariant map $\tilde{f}_s\colon |\tilde{K}|_s \to S_{-}^{d-1}$ similarly as the existence of an embedding implies the existence of the Gauss map.

It is known that the simplicial deleted product $|\tilde{K}|_s$ is equivariantly homotopic to the deleted product $|\tilde{K}|$; see [Mel09, remark below Example 3.3] and the references therein. Thus there is also a $\mathbb{Z}_2$-equivariant map $|\tilde{K}| \to S^{d-1}_-$. Therefore $|K|$ PL embeds into $\mathbb{R}^d$ by the Haefliger-Weber theorem. □

Towards a weakly injective map from topological representation.

Let $\{U_i\}$ be a good cover in $\mathbb{R}^d$ and L be the nerve of this good cover. In the following lemma we will establish the existence of a certain auxiliary map $g\colon |L| \to \mathbb{R}^d$. In order to state the properties of g, we need few preliminaries.

We say that two faces α, β in L are *remote* if there is no edge $\{a, b\} \in L$ such that $a \in \alpha$ and $b \in \beta$. We also emphasize here a certain notational issue. We recall that the vertices of L are the sets U_i. Therefore it make sense to consider the unions of faces in L. For example, if $\alpha := \{U_1, U_2\} \in L$, then

$$\bigcup \alpha = \bigcup_{U_i \in \alpha} U_i = U_1 \cup U_2.$$

Lemma 6.5. *Let $\{U_i\}$ be a good cover in $\mathbb{R}^d$ and L be its nerve. Then there is a map $g\colon |L| \to \mathbb{R}^d$ such that*

(i) $g(|\sigma|) \subseteq \bigcup \sigma$ for each $\sigma \in L$; and

(ii) $g(|\alpha|) \cap g(|\beta|) = \emptyset$ for any two remote $\alpha, \beta \in L$.

Proof. See Figure 5 while following the proof.

First we specify g on vertices of L. Then we extend it inductively to higher dimensional simplices of L.

A vertex of L is one of the sets U_i. We set $g(|\{U_i\}|)$ to be an arbitrary point inside U_i. Note that (i) is satisfied for vertices of L.

Now we inductively assume that g is defined on all simplices of L of dimension at most $k-1$. Our task is to extend g to all simplices

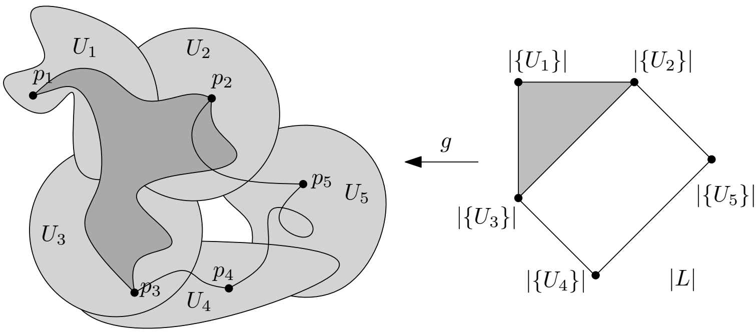

Figure 5: Map g from $|\operatorname{sd} L|$ into $\mathbb{R}^d$. The notation is slightly simplified. For example p_1 stands for $g(|\{U_1\}|)$, etc.

of L of dimension k. We also assume that condition (i) is valid for all $\sigma' \in L$ of dimension at most $k-1$.

Let σ be a k-simplex of L. From condition (i) we know that the g-images of all proper subfaces of σ belong to $\bigcup \sigma$, so $g(|\partial\sigma|) \subset \bigcup \sigma$. But $\partial\sigma$ is homeomorphic to the $(k-1)$-sphere and $\bigcup \sigma$ is contractible due to the Nerve Theorem. So we can extend g defined on $|\partial\sigma|$ to a PL map $g\colon |\sigma| \to \bigcup \sigma$. To complete the inductive step, we extend g in this way to every k-simplex $|\sigma|$. Note that condition (i) is satisfied by construction.

We have defined g so that it satisfies condition (i). It remains to show that it satisfies (ii) as well. Let α and β be remote simplices of L. By condition (i), $g(|\alpha|) \subset \bigcup \alpha$ and $g(|\beta|) \subset \bigcup \beta$. If the two right-hand unions had any intersection, this would mean there exist k, l such that $U_k \in \alpha$, $U_l \in \beta$ and $U_k \cap U_l \neq \emptyset$. But this means $\{U_k, U_l\} \in L$, so α and β are not remote. □

Proof of Theorem 6.1. Let us assume that L is some subdivision of $\operatorname{sd} K$ that is topologically d-representable. Let $\mathcal{G}$ be a topological d-representation of L. For simplicity of notation, we assume that L

is the nerve of $\mathcal{G}$. Let $g\colon |L| \to \mathbb{R}^d$ be the map from Lemma 6.5. Our task is to show that g is weakly injective with respect to K.

Let γ and δ be disjoint simplices of K. Let α be a simplex of L with $|\alpha| \subseteq |\gamma|$ and β be a simplex of L with $|\beta| \subseteq |\delta|$. Then α and β are remote in L since in particular $|\alpha| \subseteq |\gamma'|$ where γ' is some simplex of $\operatorname{sd}\gamma$ and similarly with β and $\operatorname{sd}\delta$. Thus $g(|\operatorname{sd}\alpha|) \cap g(|\operatorname{sd}\beta|) = \emptyset$ by Lemma 6.5. Consequently $g(|\gamma|) \cap g(|\delta|) = \emptyset$ for any choice of disjoint γ and δ. Therefore g is weakly injective.

We conclude by stating that Corollary 6.4 implies that K PL embeds into $\mathbb{R}^d$. □

Remark 6.6. Note that in the proof of Theorem 6.1 we only need that L is a "sufficiently fine" subdivision in the following sense: if γ and δ are disjoint simplices of K and if α and β are simplices of L satisfying $|\alpha| \subseteq |\gamma|$, and $|\beta| \subseteq |\delta|$, then α and β are remote in L. Therefore, Theorem 6.1 can be furthermore extended to such subdivisions.

7 Further questions

We have proved that for $d \geq 5$ it is algorithmically undecidable whether a given simplicial complex is topologically d-representable. In our proof we have used simplicial complexes of dimension d. It is natural to ask whether the recognition of topologically d-representable simplicial complexes becomes algorithmic if we pose some additional restrictions on these complexes.

On the positive side, there is even a polynomial algorithm deciding whether a given $d/2$-dimensional simplicial complex embeds into $\mathbb{R}^d$ (for $d \geq 6$ even, or $d = 2$). This is true because Van Kampen's obstruction is a complete obstruction for embeddability in this range and it is computable in a polynomial time; see [MTW11] for more details. Therefore by Theorems 3.3 and 6.1 we have the following corollary:

Corollary 7.1. *Let K be a simplicial complex of dimension $\frac{d}{2}$ with $d \geq 6$ even. Then there is a polynomial time algorithm deciding whether* $\operatorname{sd} K$ *is topologically d-representable.*[5]

If K is k-dimensional instead of specifically $\frac{d}{2}$-dimensional, it is in general not known whether there is an algorithm deciding whether K PL embeds into $\mathbb{R}^d$. However, based on work of Čadek et al [ČKM+12], it is plausible to believe that this embeddability question is decidable for all pairs (k, d) in the metastable range. If this is true, then Corollary 7.1 can be extended (maybe without the polynomial time estimate) to the whole metastable range.

It should be emphasized that it is quite restrictive to look for an algorithm with restrictions on the triangulation of the complex. Therefore it is natural to ask what happens if we pose only dimensional restrictions:

Question 7.2. *For which pairs of integers k and d is there an algorithm which recognizes whether a given simplicial complex of dimension at most k is topologically d-representable?*

Remark 7.3. A simplicial complex K is topologically d-representable if and only if the disjoint union of K and a simplex of arbitrary high dimension is topologically d-representable. Therefore 'at most k' can be replaced with 'exactly k' without changing the outcome.

Our main result says that the answer is no if $5 \leq d \leq k$.

If $d \geq 2k + 1$, then every simplicial complex of dimension at most k is topologically d-representable. This follows, for example, from Theorem 3.3 and the fact that every k-dimensional simplicial complex is even linearly embeddable into $\mathbb{R}^{2k+1}$.

If $d = 1$, then it is not so hard to see that the answer is yes no matter what is k, because topologically 1-representable complexes are *clique complexes over interval graphs.*

[5] Theorem 6.1 can be extended to the case $k = 1$, $d = 2$ if we use Hanani-Tutte theorem instead of Haefliger-Weber theorem. However, this is only a marginal improvement, therefore we do not include it here separately. Then we could include the case $d = 2$ in the corollary as well.

For other pairs (k, d) we do not know the answer. It would be especially interesting if there was an algorithm in the whole metastable range.

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