Quasirandom permutations are characterized by 4-point densities

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Abstract

For permutations π and τ of lengths $|\pi| \le |\tau|$, let $t(\pi, \tau)$ be the probability that the restriction of τ to a random $|\pi|$-point set is (order) isomorphic to π. We show that every sequence $\{\tau_j\}$ of permutations such that $|\tau_j| \to \infty$ and $t(\pi, \tau_j) \to 1/4!$ for every 4-point permutation π is *quasirandom* (that is, $t(\pi, \tau_j) \to 1/|\pi|!$ for every π). This answers a question posed by Graham.

1 Introduction

Roughly speaking, a combinatorial object is called *quasirandom* if it has properties that a random object has asymptotically almost

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surely. This notion has been defined for various structures such as tournaments [1], set systems [2], subsets of $\mathbb{Z}/n\mathbb{Z}$ [3], k-uniform hypergraphs [6, 7], groups [8], etc.

In particular, quasirandomness has been extensively studied for graphs. Extending earlier results of Rödl [16] and Thomason [17], Chung, Graham and Wilson [4] gave seven equivalent properties of graph sequences such that the sequence of random graphs $\{G_{n,1/2}\}$ possesses them with probability one. These properties include densities of subgraphs, values of eigenvalues of the adjacency matrix or the average size of the common neighborhood of two vertices. In particular, it follows from the results in [4] that if the density of 4-vertex subgraphs in a large graph is asymptotically the same as in $G_{n,1/2}$, then this is true for every fixed subgraph. Graham (see [5, Page 141]) asked whether a similar phenomenon also occurs in the case of permutations.

Let us state his question more precisely. Let S_k consist of permutations on $[k] := \{1, \dots, k\}$. We view each $\pi \in S_k$ as a bijection $\pi : [k] \to [k]$ and call $|\pi| := k$ its *length.* For $\pi \in S_k$ and $\tau \in S_m$ with $k \le m$, let $t(\pi, \tau)$ be the probability that a random k-point subset X of $[m]$ induces a permutation *isomorphic* to π (that is, $\tau(x_i) \le \tau(x_j)$ iff $\pi(i) \le \pi(j)$ where X consists of $x_1 < \ldots < x_k$). A sequence $\{\tau_j\}$ of permutations has *Property* $\mathbf{P}(k)$ if $|\tau_j| \to \infty$ and $t(\pi, \tau_j) = 1/k! + o(1)$ for every $\pi \in S_k$. It is easy to see that $\mathbf{P}(k+1)$ implies $\mathbf{P}(k)$. Graham asked whether there exists an integer m such that $\mathbf{P}(m)$ implies $\mathbf{P}(k)$ for every k. Here we answer this question:

Theorem 1. *Property* $\mathbf{P}(4)$ *implies Property* $\mathbf{P}(k)$ *for every* k*.*

We also give an example that $\mathbf{P}(3)$ does not imply $\mathbf{P}(4)$. (It is trivial to see that $\mathbf{P}(1) \not\Rightarrow \mathbf{P}(2)$ and an example that $\mathbf{P}(2) \not\Rightarrow \mathbf{P}(3)$ can be found in [5].)

Since these notions deal with properties of sequences of permutations, we find it convenient to operate with the appropriately defined "limit object" which is analogous to that for graphs introduced by Lovász and Szegedy [12]. Here we use the analytic aspects of permutation limits that were studied by Hoppen et al [9, 10] and we derive

Theorem 1 from its analytic analog (Theorem 3).

Let the (normalized) *discrepancy* $d(\tau)$ of $\tau \in S_n$ be the maximum over intervals $A, B \subseteq [n]$ of

$$\left| \frac{|A|\,|B|}{n^2} - \frac{|\tau(A) \cap B|}{n} \right|.$$

Cooper [5] calls a permutation sequence $\{\tau_j\}$ *quasirandom* if $|\tau_j| \to \infty$ and $d(\tau_j) \to 0$. He also gives a few other equivalent properties ([5, Theorem 3.1]) and he discusses various applications of "random-like" permutations. Using the results of [9, 10], it is not hard to relate quasirandomness and Properties $\mathbf{P}(k)$:

Proposition 2. *A sequence $\{\tau_j\}$ of permutations is quasirandom if and only if it satisfies Property $\mathbf{P}(k)$ for every k.*

Thus our Theorem 1 implies that $\mathbf{P}(4)$ alone is equivalent to quasirandomness. Finally, let us remark that McKay, Morse and Wilf [14, Page 121] also defined a notion of quasirandomness for permutations. Their definition, although related, is different from that of Cooper as it deals with sequences of *sets* of permutations.

2 Limits of permutations

Here we define convergence of permutation sequences and show how a convergent sequence can be associated with an analytic limit object. We refer the reader to [9, 10] for more details.

Let $\mathcal{Z}$ consist of probability measures μ on the Borel σ-algebra of $[0,1]^2$ that have *uniform marginals*, that is, $\mu(A \times [0,1]) = \mu([0,1] \times A) = \lambda(A)$ for every Borel set $A \subseteq [0,1]$, where λ is the Lebesgue measure on $[0,1]$.

Fix some $\mu \in \mathcal{Z}$. Let $V_i = (X_i, Y_i)$ for $i \in [k]$ be independent random variables with $V_i \sim \mu$ (that is, each V_i has distribution μ). We view an outcome $(X_1, Y_1, \ldots, X_k, Y_k)$ as an element of $[0,1]^{2k}$. For permutations $\pi, \tau \in S_k$, let $A_{\pi,\tau} \subseteq [0,1]^{2k}$ correspond to the event that

$$X_i < X_j \text{ iff } \pi(i) < \pi(j) \ \ \& \ \ Y_i < Y_j \text{ iff } \tau(i) < \tau(j).$$

Since each of the vectors $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_k)$ is uniformly distributed over $[0,1]^k$, the probability of the *degenerate event*

$$D_k := \big\{X_i = X_j \text{ or } Y_i = Y_j \text{ for some } i \neq j\big\} \subseteq [0,1]^{2k} \qquad (1)$$

is zero. Note that the sets $A_{\pi,\tau}$ for $\pi, \tau \in S_k$ partition $[0,1]^{2k} \setminus D_k$. If we reorder the indices in an outcome $(V_1, \ldots, V_k) \in [0,1]^{2k} \setminus D_k$ so that $X_1 < \ldots < X_k$, then the new relative order on $Y_1, \ldots, Y_k \in [0,1]$ defines a random permutation $\sigma(k, \mu) \in S_k$. In other words, if we land in $A_{\pi,\tau}$, then we set $\sigma(k, \mu) = \tau\pi^{-1}$. Let the *density* $t(\pi, \mu)$ of $\pi \in S_k$ be the probability that $\sigma(k, \mu) \cong \pi$. Equivalently,

$$t(\pi, \mu) = \sum_{\rho \in S_k} \mu^k(A_{\rho,\pi\rho}) = k!\, \mu^k(A_{\tau,\pi\tau}), \quad \text{any } \tau \in S_k, \qquad (2)$$

where the last equality uses the fact that $\mu^k(A_{\rho,\pi\rho})$ does not depend on $\rho \in S_k$ (because $V_1, \ldots, V_k$ are independent and identically distributed).

A sequence of permutations $\{\tau_j\}$ is *convergent* if $|\tau_j| \to \infty$ and $\{t(\pi, \tau_j)\}$ converges for every permutation π. This is the same definition of convergence as the one in [9, 10] except we additionally require that $|\tau_j| \to \infty$, cf. [9, Claim 2.4].

It is easy to show that every sequence of permutations whose lengths tend to infinity has a convergent subsequence, see e.g. [10, Lemma 2.11]. Furthermore, for every convergent sequence $\{\tau_j\}$ there is $\mu \in \mathcal{Z}$ such that for every permutation π we have

$$\lim_{j \to \infty} t(\pi, \tau_j) = t(\pi, \mu). \qquad (3)$$

For reader's convenience, we sketch the proof from [9] that μ exists. For $\pi \in S_k$, let $\mu_\pi \in \mathcal{Z}$ be obtained by dividing the square $[0,1]^2$ in $k \times k$ equal squares and distributing the mass uniformly on the squares with indices $(i, \pi(i))$, $i = 1, \ldots, k$. By Prokhorov's theorem, $\{\mu_{\tau_j}\}$ has a subsequence that weakly converges to some measure μ. We have $\mu \in \mathcal{Z}$ as this set is closed in the weak topology. Finally, μ satisfies (3) because, for any fixed π, the function $t(\pi, -) : \mathcal{Z} \to \mathbb{R}$ is continuous in the weak topology and $t(\pi, \tau_j) = t(\pi, \mu_{\tau_j}) + O(1/|\tau_j|)$.

We remark that Hoppen et al. [9, 10] proposed a slightly different limit object: the regular conditional distribution function of Y with respect to X, where $(X, Y) \sim \mu$. Lemma 2.2 and Definition 2.3 in [9] show how to switch back and forth between the two objects.

Now, we are ready to state the analytic version of Theorem 1. Let us call $\mu \in \mathcal{Z}$ *k-symmetric* if $t(\pi, \mu) = 1/k!$ for every $\pi \in S_k$.

Theorem 3. *Every 4-symmetric $\mu \in \mathcal{Z}$ is the (uniform) Lebesgue measure on $[0,1]^2$. In particular, μ is k-symmetric for every k.*

Let us show how Theorem 3 implies Theorem 1. Suppose on the contrary that some $\{\tau_j\}$ satisfies $\mathbf{P}(4)$ but not $\mathbf{P}(k)$. Fix $\pi \in S_k$ and a subsequence $\{\tau_j'\}$ such that $\lim_{j\to\infty} t(\pi, \tau_j')$ exists and is not equal to $1/k!$. Consider now a convergent subsequence $\{\tau_j''\}$ of $\{\tau_j'\}$ and let $\mu \in \mathcal{Z}$ be its limit. By (3), μ is 4-symmetric and, by Theorem 3, μ is m-symmetric for every m. But then $\lim_{j\to\infty} t(\pi, \tau_j'') = t(\pi, \mu) = 1/k!$, which is the desired contradiction.

3 Proof of Theorem 3

In this section, let $\mu \in \mathcal{Z}$ be arbitrary with $t(\pi, \mu) = 1/4!$ for every $\pi \in S_4$. Let $\lambda \in \mathcal{Z}$ denote the uniform measure on $[0,1]^2$. Our objective is to show that $\mu = \lambda$.

Let $V = (X, Y) \sim \mu$ and $v = (x, y) \sim \lambda$ be independent. For brevity, let us abbreviate $\int_{[0,1]^2}$ to $\int$. Define a function $F : [0,1]^2 \to [0,1]$ by

$$F(a, b) := \mu([0,a] \times [0,b]) = \int_{V \le (a,b)} \mathrm{d}V,$$

where $V \le (a, b)$ means that $X \le a$ and $Y \le b$. Since μ has uniform marginals, the function F is continuous.

First, we show that the 4-symmetry of μ uniquely determines certain integrals.

Lemma 4.

$$\int F(X,Y)^2 \, \mathrm{d}V = \int F(X,Y)XY \, \mathrm{d}V = \int F(x,y)^2 \, \mathrm{d}v = \frac{1}{9}.$$

Proof. Let $V_i = (X_i, Y_i) \sim \mu$, for $i = 1, 2, \ldots$, be independent random variables distributed according to μ. By Fubini's theorem, we have

$$\int F(X,Y)^2 \, \mathrm{d}V = \int \left(\int_{V_2 \le V_1} \mathrm{d}V_2 \right) \left(\int_{V_3 \le V_1} \mathrm{d}V_3 \right) \mathrm{d}V_1 = \int_A \mathrm{d}(V_1, V_2, V_3),$$

where $A = \{(V_1, V_2, V_3) : V_2 \le V_1 \,\&\, V_3 \le V_1\} \subseteq [0,1]^6$. Note that

$$A \setminus D_3 = \bigcup_{\substack{\pi, \tau \in S_3 \\ \pi(1) = \tau(1) = 3}} A_{\pi,\tau},$$

where D_3 is defined by (1). The 4-symmetry of μ and (2) imply that $\mu^k(A_{\pi,\tau}) = (1/k!)^2$ for every $k \le 4$ and $\pi, \tau \in S_k$. Since $\mu^3(D_3) = 0$, we have $\mu^3(A) = 4 \cdot (1/3!)^2 = 1/9$, as required.

Likewise, $\int F(X,Y)XY \, \mathrm{d}V = \int_B \mathrm{d}(V_1, \ldots, V_4)$, where $B \subseteq [0,1]^8$ corresponds to the event that $V_2 \le V_1$, $X_3 \le X_1$ and $Y_4 \le Y_1$. The latter integral is equal to the μ^4-measure of the union of $A_{\pi,\tau}$ over some (explicit) set of pairs $\pi, \tau \in S_4$. The measure of this set is uniquely determined by the 4-symmetry of μ. Thus the integral does not change if we replace μ by any other 4-symmetric measure. Considering the uniform measure λ, we obtain $\int x^2 y^2 \, \mathrm{d}v = 1/9$, as required.

Next, observe that (X_1, Y_2) is uniformly distributed in $[0,1]^2$ because V_1 and V_2 are independent and have uniform marginals. Again, the value of

$$\int F(x,y)^2 \, \mathrm{d}v = \int_{[0,1]^4} F(X_1, Y_2)^2 \, \mathrm{d}(V_1, V_2) = \int_{V_3, V_4 \le (X_1, Y_2)} \mathrm{d}(V_1, \ldots, V_4),$$

does not depend on the choice of μ and can be easily computed by taking $\mu = \lambda$. $\square$

Since X is uniformly distributed in $[0,1]$, we have $\int X^2 \, \mathrm{d}V = 1/3$. Also,

$$\int F(x,y)xy \, \mathrm{d}v = \int_{v \geq V} xy \, \mathrm{d}(v,V) = \frac{1}{4}\int (1 - X^2 - Y^2 + X^2Y^2) \, \mathrm{d}V.$$

We use the above identities and apply the Cauchy-Schwartz inequality twice to get the following series of inequalities:

$$\begin{aligned}
\frac{1}{81} &= \left(\int F(X,Y)XY \, \mathrm{d}V\right)^2 \leq \left(\int F(X,Y)^2 \, \mathrm{d}V\right) \cdot \left(\int X^2Y^2 \, \mathrm{d}V\right) \\
&= \frac{1}{9}\left(4 \cdot \int F(x,y)xy \, \mathrm{d}v - \int (1 - X^2 - Y^2) \, \mathrm{d}V\right) \\
&= \frac{1}{9}\left(4 \cdot \int F(x,y)xy \, \mathrm{d}v - \frac{1}{3}\right) \\
&\leq \frac{4}{9}\sqrt{\int F(x,y)^2 \, \mathrm{d}v} \cdot \sqrt{\int x^2y^2 \, \mathrm{d}v} - \frac{1}{27} = \frac{1}{81}.
\end{aligned}$$

Thus we have equality throughout. However, the last inequality is equality if and only if $F(a,b)$ is equal to a fixed multiple of ab almost everywhere with respect to the uniform measure λ. Since F is continuous, we conclude that $F(a,b) = ab$ for all $(a,b) \in [0,1]^2$. Thus the measures μ and λ coincide on all rectangles $[0,a] \times [0,b]$. Since rectangles generate the Borel σ-algebra on $[0,1]^2$, we have that $\mu = \lambda$ by the uniqueness statement of the Carathéodory Theorem. This proves Theorem 3.

Remark 5. Our proof gives other sufficient conditions for $\mu = \lambda$. For example, it suffices to require that each of the three integrals of Lemma 4 is $1/9$. The proof of the lemma shows that, if desired, these integrals can be expressed as linear combinations of densities $t(\pi,\mu)$ for $\pi \in S_4$. The single identity $(\int F(x,y)xy \, \mathrm{d}v)^2 = \frac{1}{9}\int F(x,y)^2 \, \mathrm{d}v$ is also sufficient for proving that $\mu = \lambda$; however, if written as a polynomial in terms of permutation densities, it involves 5-point permutations. Our method can give other sufficient conditions in this manner; the choice of which one to use may depend on the available information about the sequence.

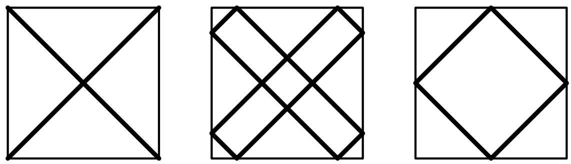

Figure 1: The sets $M(0)$, $M(1/3)$ and $M(1)$.

Remark 6. Also, the argument of Lemma 4 shows that, for every polynomial $P(x,y)$ and $\mu \in \mathcal{Z}$, the value of $\int P(x,y)\, \mathrm{d}\mu(x,y)$ can be expressed a linear combination of permutation densities. This observation combined with the Stone-Weierstrass Theorem gives the *uniqueness* of a permutation limit: if $\mu, \mu' \in \mathcal{Z}$ have the same permutation densities, then $\mu = \mu'$ (cf. [9, Theorem 1.7]).

4 P(3) does not imply P(4)

First, we construct a 3-symmetric measure $\mu \in \mathcal{Z}$ which is not 4-symmetric. For $a \in [0,1]$, let $M(a)$ be the set of all the points $(x,y) \in [0,1]^2$ such that $x + y \in \{1 - a/2, 1 + a/2, a/2, 2 - a/2\}$ or $y - x \in \{-a/2, a/2, 1 - a/2, a/2 - 1\}$. See Figure 1 for illustrations of this definition. Define $\mu_a \in \mathcal{Z}$ for $a \in [0,1]$ to be the permutation limit such that the mass is uniformly distributed on $M(a)$. Because of the symmetries of μ_a (invariance under the horizontal and vertical reflections), we have that $t(\pi, \mu_a) = 1/6$ for every $\pi \in S_3$ if and only if $t(\mathrm{Id}_3, \mu_a) = 1/6$, where Id_3 is the identity 3-point permutation.

Routine calculations show that $t(\mathrm{Id}_3, \mu_0) = 1/4$ and $t(\mathrm{Id}_3, \mu_1) = 1/8$. Since $t(\mathrm{Id}_3, \mu_a)$ is continuous in a, there exists $b \in [0,1]$ such that $t(\mathrm{Id}_3, \mu_b) = 1/6$. Moreover, μ_b is not 4-symmetric. This can be verified directly; it also follows from Theorem 3 since μ_b is not the uniform measure.

Take a sequence $\{\tau_j\}$ of permutations that converges to μ_b. For example, the random sequence $\{\sigma(j, \mu_b)\}$ has this property with probability one, see [10, Corollary 4.3]. Any such sequence $\{\tau_j\}$ satisfies $\mathbf{P}(3)$ but not $\mathbf{P}(4)$.

5 Proof of Proposition 2

Let $\{\tau_j\}$ be an arbitrary sequence of permutations with $|\tau_j| \to \infty$. Let $\mu_j \in \mathcal{Z}$ be the measure associated with τ_j as is described after (3). It is straightforward to verify that $d(\tau_j) = d(\mu_j) + o(1)$, where

$$d(\mu) := \sup \left|\lambda(A \times B) - \mu(A \times B)\right|$$

denotes the *discrepancy* of $\mu \in \mathcal{Z}$, with the supremum (in fact, it is maximum) being taken over intervals $A, B \subseteq [0,1]$. The uniqueness of a permutation limit (see Remark 6) implies that $\{\tau_j\}$ converges to μ if and only if $\{\mu_j\}$ weakly converges to μ.

First, suppose that $\{\tau_j\}$ satisfies $\mathbf{P}(k)$ for each k. This means that $\{\tau_j\}$ converges to the uniform limit λ. For $a, b \in [0,1]$, let $F_j(a,b) := \mu_j([0,a] \times [0,b])$ and $F(a,b) := ab$. Since $d(\lambda) = 0$ and

$$\mu_j([a_1,a_2] \times [b_1,b_2]) = F_j(a_2,b_2) - F_j(a_1,b_2) - F_j(a_2,b_1) + F_j(a_1,b_1),$$

we conclude that $d(\mu_j) \leq 4 \cdot \|F_j - F\|_\infty$. The weak convergence $\mu_j \to \lambda$ of measures in $\mathcal{Z}$ gives that $F_j \to F$ pointwise. Since F and each function F_j are continuous and monotone in both coordinates, this implies that

$$\|F_j - F\|_\infty \to 0. \tag{4}$$

Thus $d(\mu_j) \to 0$ and $\{\tau_j\}$ is quasirandom. (Alternatively, (4) directly follows from [9, Lemma 5.3].)

Next suppose that $d(\tau_j) \to 0$. One way to establish Property $\mathbf{P}(k)$ is to use one of the equivalent definitions of quasirandomness from [5, Theorem 3.1] (namely Property [mS]). Alternatively, if $\mathbf{P}(k)$ fails, then (by passing to a subsequence) we can assume that $\{\tau_j\}$ converges to some $\mu \in \mathcal{Z}$ with $\mu \neq \lambda$. However, it holds that $d(\mu) = 0$ which implies $\mu = \lambda$, contradicting our assumption. This finishes the proof of Proposition 2.

6 Concluding remarks

The theory of flag algebras developed by Razborov [15] can be applied to permutation limits: a permutation $\pi : A \to A$ is viewed as two binary relations, each giving a linear order on A. For example, Lemma 4 can be stated and proved within the flag algebra framework. This view has been very helpful for us when developing our proof.

A graph can be associated with a permutation $\pi \in S_n$ as follows: let $G(\pi)$ be the graph on $[n]$ with vertices $i < j$ adjacent if $\pi(i) < \pi(j)$. Fix $\mu \in \mathcal{Z}$ and sample a random permutation $\sigma(n, \mu)$. Define a function $W : [0,1]^4 \to \{0,1\}$ by $W(x_1, y_1, x_2, y_2) = 1$ if we have $(x_1, y_1) < (x_2, y_2)$ or $(x_1, y_1) > (x_2, y_2)$ componentwise and let $W(x_1, y_1, x_2, y_2) = 0$ otherwise. In other words, W is the indicator function of the event that $\sigma(2, \mu)$ is the identity 2-point permutation. Clearly, $G(\sigma(n, \mu))$ can be generated by sampling independently points $V_1, \dots, V_n \in [0,1]^2$, each with distribution μ, and connecting those $i, j \in [n]$ for which $W(V_i, V_j) = 1$. The latter procedure corresponds to generating a random sample $\mathbb{G}(n, W)$, where W is viewed as a graphon represented on Borel subsets of $[0,1]^2$ with measure μ, see [12, Section 2.6] for details.

Lovász and Sós [11] and Lovász and Szegedy [13] presented various sufficient conditions for a graphon W to be *finitely forcible* which, in the above notation, means that there is m such that the distribution of $\mathbb{G}(m, W)$ uniquely determines that of $\mathbb{G}(k, W)$ for every k. As far as we can see, none of these conditions directly applies to the graphon associated with the uniform measure $\lambda \in \mathcal{Z}$. Since we answered Graham's question on quasirandom permutations by other means, we did not pursue this approach any further.

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