Extending fractional precolorings *

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Abstract

For every $d \geq 3$ and $k \in \{2\} \cup [3, \infty)$, we determine the smallest ε such that every fractional $(k+\varepsilon)$-precoloring of vertices at mutual distance at least d of a graph G with fractional chromatic number equal to k can be extended to a proper fractional $(k+\varepsilon)$-coloring of G. Our work complements the analogous results of Albertson for ordinary colorings and those of Albertson and West for circular colorings.

1 Introduction

One of the most major results in graph theory is the Four Color Theorem which asserts that every planar graph is 4-colorable. In [13], Thomassen posed the following problem.

Problem 1. *Suppose that G is a plane graph and W a subset of its vertices such that the distance between any two vertices of W is at least 100. Can a 5-coloring of W be extended to a 5-coloring of G?*

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Shortly after that, the problem was answered in affirmative by Albertson [1] who established the following theorem.

Theorem 2. *Let G be an r-colorable graph and W a subset of its vertices such that the distance between any two vertices of W is at least four. Every $(r+1)$-coloring of W can be extended to an $(r+1)$-coloring of G.*

It is possible to show that the constraint on the mutual distance between the vertices of W to be at least four in Theorem 2 is the best possible even if the graph G is planar, see [1] for further details.

Theorem 2 initiated a line of research [2–6, 9] where necessary and sufficient conditions for the existence of an extension of a coloring of various types of subgraphs have been considered. Inspired by this line of research, Albertson and West [7] considered the coloring extension problem in the circular coloring setting. The notion of circular colorings is a well-known relaxation of classical colorings introduced in [14]; a wide list of results on circular colorings can be found in two surveys on this notion by Zhu [15, 16]. Another well-established relaxation of classical colorings is the notion of fractional colorings, see [12], which we address in this paper.

Before we state our results, let us introduce the related notation. A *fractional k-coloring* of a graph G is an assignment of measurable subsets of the interval $[0, k)$ to the vertices of G such that each vertex receives a subset of measure one and adjacent vertices receive disjoint subsets. The *fractional chromatic number of* G is the smallest k such that G admits a fractional k-coloring; it can be shown that such k exists (the minimum is attained) and the value of k is always a rational (if G is finite). There is another (more discrete) definition of the fractional chromatic number: the fractional chromatic number of a graph G is the minimum of p/q such that there exists a coloring of the vertices of G with p colors such that every vertex of G is assigned q colors and adjacent vertices are assigned disjoint sets of colors. Another definition of the fractional chromatic number can be given through a linear program assigning weights to independent sets of G.

In this paper, we study conditions under which a fractional coloring of a part of a graph can be completed to a fractional coloring of the whole graph. To be precise, call a *fractional k-precoloring* an assignment of measurable subsets of the interval $[0, k)$ of unit measure to some vertices of a graph. If the precolored vertices can be at distance at most two, then there is no k such that any fractional k-precoloring can be extended to a fractional k-coloring of G even if we assume that G is a tree. Hence, we restrict our attention to precolorings of vertices at distance $d \geq 3$.

Analogously to Albertson and West [7], we study what is the minimum value of ε such that any fractional $(k+\varepsilon)$-precoloring of vertices at mutual distance d of a fractionally k-colorable graph can be extended to a fractional $(k+\varepsilon)$-coloring of the whole graph. Since the fractional chromatic number and the fractional list chromatic number of a graph are always equal [8] (also see [12, Theorem 3.8.1]), it follows that any such precoloring can be extended for $\varepsilon \geq 1$ if $d \geq 3$; thus, the minimum value is always at most one.

In general, the sought minimum value of ε depends on k and d. For $k \in \{2\} \cup [3,\infty)$ and any $d \geq 3$ (see our previous discussion for $d \leq 2$), we have been able to determine the optimal minimum value. For $k \in (2,3)$, upper bounds are given. Our findings are summarized in the next theorem.

Theorem 3. *Let G be a graph with fractional chromatic number $k \in [2,\infty)$ and W a subset of its vertices at mutual distance at least $d \geq 3$. For every real $\varepsilon > 0$ satisfying*

$$\begin{array}{rcll} k-1 & \leq & d'\varepsilon^2 + (d'k-1)\varepsilon & \textit{if } d = 0 \textit{ mod } 4, \\ k-1 & \leq & d'k\varepsilon & \textit{if } d = 1 \textit{ mod } 4, \\ k-1 & \leq & d'\varepsilon^2 + d'k\varepsilon & \textit{if } d = 2 \textit{ mod } 4, \textit{ and} \\ k-1 & \leq & (d'k+k-1)\varepsilon & \textit{otherwise,} \end{array} \tag{1}$$

where $d' = \lfloor d/4 \rfloor$, every fractional $(k+\varepsilon)$-precoloring of the vertices of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.

Moreover, for every rational $k \in \{2\} \cup [3,\infty)$ and every $\varepsilon > 0$ not satisfying (1), there exists a graph G with fractional chromatic k, a subset W of its vertices at mutual distance at least d and a fractional $(k+\varepsilon)$-precoloring of the vertices of W that cannot be extended to a fractional $(k+\varepsilon)$-coloring of G.

Theorem 3 follows from Theorems 9 and 12 which we establish in Sections 3 and 4. We have managed to give a construction that the condition (1) is the best possible for $k \in \{2\} \cup [3,\infty)$. The case $k \in (2,3)$ seems to be significantly more difficult as we discuss in Section 5.

2 Universal graphs

The definition of fractional colorings allows defining a class of universal graphs, i.e., a class such that every graph with fractional chromatic number k has a homomorphism to one of the graphs in this class. Recall that a *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to$

$V(H)$ such that if u and v are two adjacent vertices of G, then the vertices $f(u)$ and $f(v)$ are adjacent in H. If such a mapping exists, we say that G is *homomorphic* to H.

Universal graphs for fractional colorings are Kneser graphs $K_{p,q}$; a graph $K_{p,q}$ for integers $1 \le q \le p$ has a vertex set formed by all q-element subsets of $[p] = \{1, \dots, p\}$, i.e., $V(K_{p,q}) = \binom{[p]}{q}$. Two vertices A and A' are adjacent if $A \cap A' = \emptyset$. The fractional chromatic number of $K_{p,q}$ is clearly at most p/q, and it is not hard to show that it is actually equal to p/q. The definition of the fractional chromatic number yields the following proposition which can also be found, e.g., in [11]. We include a short proof for the sake of completeness.

Proposition 4. *Let G be a graph with fractional chromatic number k. There exist integers p and q such that $k = p/q$ and G is homomorphic to the graph $K_{p,q}$.*

Proof. By the definition of the fractional chromatic number, there exist integers p and q, $k = p/q$, and a mapping $f : V(G) \to \binom{[p]}{q}$ such that $f(u) \cap f(v) = \emptyset$ for any two adjacent vertices u and v of G (consider a coloring with p colors assigning adjacent vertices disjoint sets of q colors). Observe that the mapping f is a homomorphism from G to $K_{p,q}$. $\square$

Our proofs are based on defining and analysing graphs that are universal for graphs (of a given fractional chromatic) with some vertices precolored. Graphs we now define are sketched in Figures 1–3.

For integers p, q and ℓ, $1 \le q \le p/2$ and $1 \le \ell$, and a subset $X \in \binom{[p]}{q}$, the *ray* $S^X_{p,q,\ell}$ is the graph with vertex set formed by all pairs (A, i) where $A \in \binom{[p]}{q}$ and $1 \le i \le \ell$ and the pair $(X, 0)$. Two vertices (A, i) and (A', i') are adjacent if $A \cap A' = \emptyset$ and $|i - i'| \le 1$. For brevity, $S_{p,q,\ell}$ will stand for $S^{[q]}_{p,q,\ell}$ in what follows. The vertex $(X, 0)$ of $S^X_{p,q,\ell}$ is called the *special vertex* of $S^X_{p,q,\ell}$.

Observe that the graph $S^X_{p,q,\ell}$ is homomorphic to $K_{p,q}$: map a vertex (A, i) of $S^X_{p,q,\ell}$ to the vertex A of $K_{p,q}$. In addition, the distance of the vertex $(X, 0)$ from any vertex (A, ℓ), $A \in \binom{[p]}{q}$, is at least ℓ and the subgraph of $S^X_{p,q,\ell}$ induced by vertices (A, i), $A \in \binom{[p]}{q}$, for a fixed integer i, $1 \le i \le \ell$, is isomorphic to $K_{p,q}$. The subgraph of $S_{p,q,\ell}$ formed by vertices (A, ℓ), $A \in \binom{[p]}{q}$, is called the *base* of $S_{p,q,\ell}$.

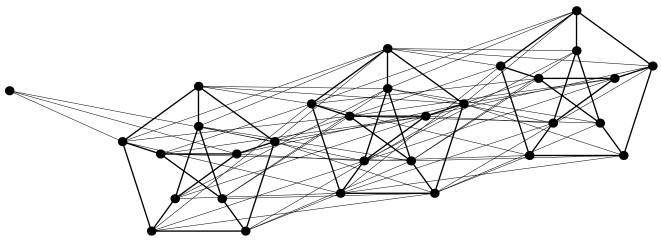

Figure 1: The ray $S_{5,2,3}$.

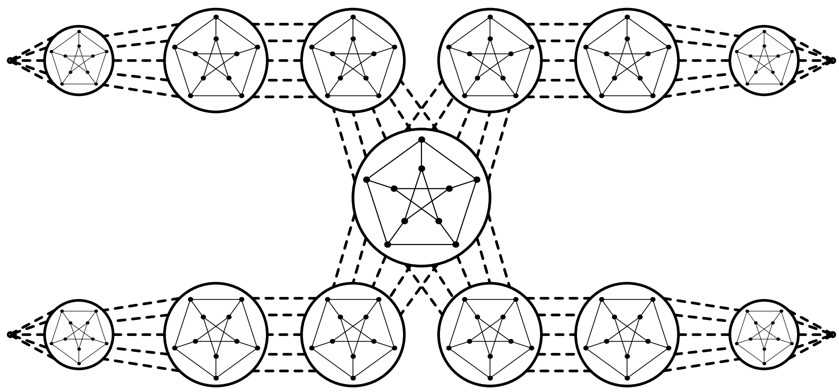

Figure 2: A sketch of the graph $P^1_{5,2,8}$ (some rays are omitted).

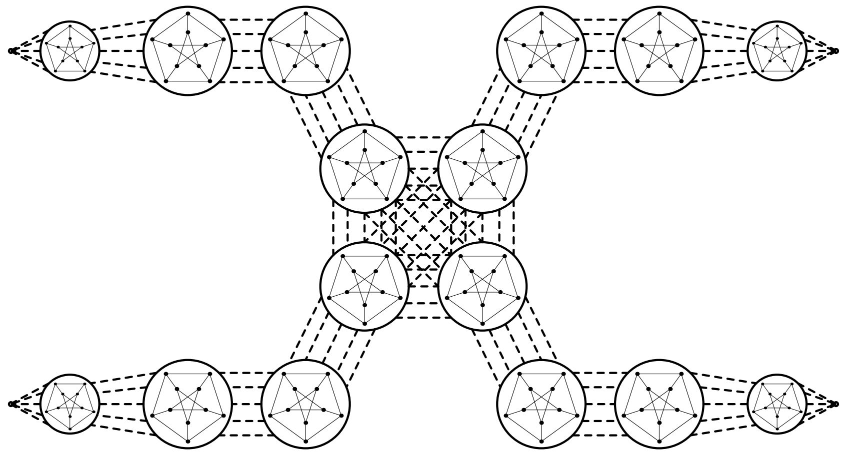

Figure 3: A sketch of the graph $P^1_{5,2,9}$ (some rays are omitted).

The graph $P^n_{p,q,d}$, which we now define, is a universal graph for graphs with fractional chromatic number p/q with precolored vertices at distance d. Fix integers $1 \le q \le p/2$, $3 \le d$ and $1 \le n$. If d is even, the graph $P^n_{p,q,d}$ is obtained by taking n copies of each of the rays $S^X_{p,q,d/2}$ for every choice of $X \in \binom{[p]}{q}$ and identifying the vertices $(A, d/2)$ in all the copies to a single vertex. The $n\binom{p}{q}$ special vertices of the rays used in the construction are referred to as *special vertices* of $P^n_{p,q,d}$.

If d is odd, the graph $P^n_{p,q,d}$ is obtained by taking n copies of each of the rays $S^X_{p,q,(d-1)/2}$ for every choice of $X \in \binom{[p]}{q}$ and joining any two vertices $(A, (d-1)/2)$ and $(A', (d-1)/2)$ (of different copies) by an edge if $A \cap A' = \emptyset$. Again, the special vertices of the rays are referred to as special vertices of $P^n_{p,q,d}$.

In the next two propositions, we summarize the properties of the graphs $P^n_{p,q,d}$ needed further. The proof of the first one is straightforward and we leave it to the reader.

Proposition 5. *The graph $P^n_{p,q,d}$ for $1 \le q \le p/2$ and $3 \le d$ is homomorphic to $K_{p,q}$ and its special vertices are at mutual distance at least d.*

Proposition 6. *Let G be a graph with fractional chromatic number k and*

W a subset of its vertices at mutual distance at least $d \geq 3$. There exist integers p and q, $1 \leq q \leq p/2$, and $k = p/q$, such that the graph G has a homomorphism to $P^{|W|}_{p,q,d}$ that maps the vertices of W to the mutually distinct special vertices.

Proof. Let $n = |W|$ and $W = \{w_1, \ldots, w_n\}$. By the definition of a fractional coloring, there exist integers p and q and a mapping $f : V(G) \to \binom{[p]}{q}$ such that $f(u) \cap f(v) = \emptyset$ for any two adjacent vertices u and v of G. We show that G has a homomorphism to $P^n_{p,q,d}$ as described in the statement of the proposition.

Let $S^{X,i}_{p,q,\ell}$, $X \in \binom{[p]}{q}$, $\ell = \lfloor d/2 \rfloor$ and $1 \leq i \leq n$, be the i-th copy of $S^X_{p,q,\ell}$ used in the construction of $P^n_{p,q,d}$. The homomorphism from G to $P^n_{p,q,d}$ is defined as follows. A vertex v of G at distance at most $\lfloor (d-1)/2 \rfloor$ from a vertex w_i of W is mapped to the vertex $(f(v), d')$ of the copy $S^{f(w_i),i}_{p,q,\ell}$ where d' is the distance of v from w_i. In particular, if $v = w_i$, then v is mapped to the special vertex of $S^{f(w_i),i}_{p,q,\ell}$. Hence, all vertices of W are mapped to distinct special vertices of $P^n_{p,q,d}$.

Since no vertex v can be at distance at most $\lfloor (d-1)/2 \rfloor$ from two different vertices of W (the mutual distance between vertices of W is d), the mapping is well-defined for vertices of G at distance at most $\lfloor (d-1)/2 \rfloor$ from a vertex of W. If v is a vertex at distance at least $\lceil d/2 \rceil$ from all vertices of W, then v is mapped to the vertex $f(v), \ell)$ of an arbitrary ray, e.g., to that of $S^{[q],1}_{p,q,\ell}$.

It is straightforward to verify that the just defined mapping is a homomorphism from G to the graph $P^n_{p,q,d}$. $\square$

3 Upper bounds

In this section, we focus on proving that every precoloring satisfying the assumptions of Theorem 3 can be extended. Before we proceed with detailed presentation of our arguments, let us explain the main ideas we apply. By Proposition 6, it is enough to consider universal graphs $P^n_{p,q,d}$. For a given precoloring of the special vertices, we first color the vertices of the base in such a way that their coloring is pseudorandom with respect to the special vertices. This means that if d is even, then the measure of the intersection of the set assigned to any vertex in the base and the set of any special vertex is $1/(k+\varepsilon)$ (which would be the expected intersection if we drew a random unit interval). If d is odd, we require that the measure of the intersection of the set of any special vertex and the union of the sets assigned to the copies of the same vertex in the bases of the rays is $1/k$. The pseudorandomness of the coloring of the base guarantees that the coloring behaves uniformly with respect to all the rays forming $P^n_{p,q,d}$. This coloring of the base and the special vertices is eventually extended inside the rays.

We start our exposition with two lemmas on extending fractional precolorings of the special vertex and the base inside a ray. The first one will be applied if $\lfloor d/2 \rfloor$ is even and the second one if $\lfloor d/2 \rfloor$ is odd. To state the lemmas we need additional notation. For a set $A \subseteq [0,\infty)$, 2^A stands for the set of all measurable subsets of A and $||A||$ denotes the measure of A. We write $f : [p] \hookrightarrow 2^A$ for mappings from $[p]$ to 2^A such that $f(i) \cap f(j) = \emptyset$ for any two distinct integers $i, j \in [p]$. If B is a subset of $[p]$, then $f(B)$ is the union of $f(b)$ with $b \in B$.

We can now state the first of the two lemmas.

Lemma 7. *Let p, q and ℓ be integers, $1 \le q \le p/2$, $\ell = 0 \bmod 2$ and $\ell \ge 2$, and $\varepsilon > 0$. Let $f : [p] \hookrightarrow 2^{[0,p/q+\varepsilon)}$ be a mapping such that $||f(i)|| = 1/q$ for every $i \in [p]$ and let C be a subset of $[0, p/q+\varepsilon)$ with $||C|| = 1$. The fractional $(p/q+\varepsilon)$-precoloring of $S_{p,q,\ell}$ given by assigning the special vertex the set C and each vertex (A,ℓ) of the base the set $f(A)$ can be extended to a proper fractional $(p/q+\varepsilon)$-coloring of $S_{p,q,\ell}$ if*

$$1 \le ||C \cap f([q])|| + \frac{\ell\varepsilon}{2} \,. \tag{2}$$

Proof. The proof proceeds by induction on ℓ. Suppose that $\ell = 2$. Since it holds that $||C \cap f([p] \setminus [q])|| \le \varepsilon$, there exists a measure preserving mapping $m : [0, p/q+\varepsilon) \to [0, p/q+\varepsilon)$ such that

- m is an identity on $f([q])$,
- m is an identity on $f([p] \setminus [q]) \setminus C$ and
- $m(f([p] \setminus [q]) \cap C) \cap C = \emptyset$.

Assigning each vertex $(A, 1)$, $A \in \binom{[p]}{q}$, the set $m(f(A))$ yields a proper fractional $(p/q+\varepsilon)$-coloring of $S_{p,q,2}$.

We now assume that $\ell \ge 4$. Let C_0 be a subset of $C \cap f([p] \setminus [q])$ with $||C_0|| = \max\{\varepsilon, ||C \cap f([p] \setminus [q])||\}$. Consider now a measure preserving mapping $m_1 : [0, p/q+\varepsilon) \to [0, p/q+\varepsilon)$, which exists since $||C_0|| \le \varepsilon$, such that

- m_1 is an identity on $f([q])$,

- m_1 is an identity on $f([p] \setminus [q]) \setminus C_0$, and
- $m_1(C_0) \cap C = \emptyset$.

Further, let $m_2 : [0, p/q+\varepsilon) \to [0, p/q+\varepsilon)$ be a measure preserving mapping (again, its existence follows from the fact that $||C_0|| \le \varepsilon$) such that

- $(f([q]) \cap C) \cup C_0 \subseteq m_2(f([q]))$ and
- m_1 and m_2 coincide on $f([p]) \setminus m_2^{-1}(C_0) \supseteq f([p] \setminus [q])$.

We now extend the precoloring to the vertices $(A, \ell-2)$ and $(A, \ell-1)$, $A \in \binom{[p]}{q}$. The vertex $(A, \ell-2)$ is assigned the set $m_2(f(A))$ and the vertex $(A, \ell-1)$ the set $m_1(f(A))$. It is straightforward to verify that the obtained fractional precoloring assigns adjacent vertices disjoint subsets of $[0, p/q+\varepsilon)$. Moreover, the set assigned to $([q], \ell-2)$ now contains $(f([q]) \cap C) \cup C_0$, i.e., it intersects C on a set of measure at least $1 - (\ell-2)\varepsilon/2$. By induction, the precoloring can be extended to the rest of the ray. □

Lemma 8. *Let p, q and ℓ be integers, $1 \le q \le p/2$, $\ell = 1$ mod 2 and $\ell \ge 1$, and $\varepsilon > 0$. Let $f : [p] \hookrightarrow 2^{[0,p/q+\varepsilon)}$ be a mapping such that $||f(i)|| = 1/q$ for every $i \in [p]$ and let C be a subset of $[0, p/q+\varepsilon)$ with $||C|| = 1$. The fractional $(p/q+\varepsilon)$-precoloring of $S_{p,q,\ell}$ given by assigning the special vertex the set C and each vertex (A, ℓ) of the base the set $f(A)$ can be extended to a proper fractional $(p/q+\varepsilon)$-coloring of $S_{p,q,\ell}$ if*

$$||C \cap f([p] \setminus [q])|| \le \frac{(\ell-1)\varepsilon}{2} . \tag{3}$$

If $\ell = 1$, we further require that $f([p] \setminus [q]) \cap C = \emptyset$.

Proof. If $\ell = 1$, then the statement holds since the precoloring is a proper coloring of all the vertices of $S_{p,q,\ell}$. Hence, assume $\ell \ge 3$. Let $C_0 = C \setminus f([p])$ and let $m : f([q]) \cup C_0 \to f([q]) \cup C_0$ be a measure preserving mapping such that $m^{-1}(C_0) \subseteq f([q]) \setminus C$ and m is an identity on $f([q]) \setminus m^{-1}(C_0)$. Define a mapping $f' : [p] \hookrightarrow 2^{[0,p/q+\varepsilon)}$ by setting $f'(i) = m(f(i))$ for $i \le q$ and $f'(i) = f(i)$ for $i > q$; then color a vertex $(A, \ell-1)$ of $S_{p,q,\ell}$ by $f'(A)$.

By the choice of f', the sets assigned to adjacent vertices are disjoint. Observe that $f'([q]) \cap C = (f([q]) \cap C) \cup C_0$ and thus $||f'([q]) \cap C|| = 1 - ||C \cap f([p] \setminus [q])||$. Since $1 \le ||C \cap f'([q])|| + (\ell-1)\varepsilon/2$, the precoloring of the vertices $(A, \ell-1)$ can be extended to a proper coloring by Lemma 7. □

We are now ready to prove that the conditions given in Theorem 3 guarantee the existence of an extension of a precoloring.

Theorem 9. *Let G be a graph with fractional chromatic number $k \in [2, \infty)$ and W a subset of its vertices at mutual distance at least $d \geq 3$. For $\varepsilon > 0$ satisfying (1), every fractional $(k+\varepsilon)$-precoloring of the vertices of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Proof. By Proposition 6, it is enough to prove the theorem for graphs $P^n_{p,q,d}$ with special vertices precolored. Let $C_1, \ldots, C_{n\binom{p}{q}}$, be the subsets of $[0, p/q+\varepsilon)$ assigned by the precoloring to the special vertices of $P^n_{p,q,d}$. Define an equivalence relation on the points of $[0, p/q+\varepsilon)$ such that two points x and y are equivalent if $|\{x, y\} \cap C_i| \in \{0, 2\}$ for every $i = 1, \ldots, n\binom{p}{q}$. Let $D_1, \ldots, D_m$ be the equivalence classes of this relation. Observe that $m \leq 2^{n\binom{p}{q}}$ and every set D_i, $i \in [m]$, is measurable.

The proof now proceeds differently depending on the parity of d. Suppose first that d is even. Partition each set D_i, $1 \leq i \leq m$, into $p+1$ subsets $D^0_i, \ldots, D^p_i$ such that $||D^0_i|| = \frac{\varepsilon||D_i||}{k+\varepsilon}$ and $||D^1_i|| = \cdots = ||D^p_i|| = \frac{||D_i||}{q(k+\varepsilon)}$ and define a mapping $f : [p] \hookrightarrow 2^{[0,p/q+\varepsilon)}$ by setting $f(j) = \bigcup_{i=1}^m D^j_i$, $j \in [p]$. Observe that the sets $f(1), \ldots, f(p)$ are disjoint and each has measure $1/q$. The construction of the sets $D_1, \ldots, D_m$ implies that $||C_i \cap f(j)|| = 1/(p+q\varepsilon)$ for every $i \in [n\binom{p}{q}]$ and $j \in [p]$.

We now color the vertices of the bases of the rays used in the construction of $P^n_{p,q,d}$; the vertex $(A, d/2)$ of $P^n_{p,q,d}$, $A \in \binom{[p]}{q}$, is assigned the set $f(A)$. Since the sets $f(1), \ldots, f(p)$ are disjoint and $||f(1)|| = \cdots = ||f(p)|| = 1/q$, we have obtained a proper precoloring of the base and the special vertex in each ray. Consider one of the rays, say $S^X_{p,q,d}$, and let C_i be the subset assigned to its special vertex. Since $||C_i \cap f(j)|| = 1/(p+q\varepsilon)$ for every $j \in [p]$, it holds that

$$||C_i \cap f(X)|| = \frac{1}{k+\varepsilon} \qquad \text{and} \qquad ||C_i \cap f([p] \setminus X)|| = \frac{k-1}{k+\varepsilon}\ .$$

If $d/2$ is even, the coloring can be extended to the ray by Lemma 7, and if $d/2$ is odd, it can be extended by Lemma 8. This finishes the analysis of the case when d is even.

In the remainder of the proof, we assume that d is odd. In this case, we split each D_i, $1 \leq i \leq m$, into p subsets $D^1_i, \ldots, D^p_i$ such that $||D^1_i|| = \cdots = ||D^p_i|| = ||D_i||/p$. Next define a mapping $f : [p] \hookrightarrow 2^{[0,p/q+\varepsilon)}$ by setting

$f(j) = \bigcup_{i=1}^{m} D_i^j$, $j \in [p]$. Observe that the sets $f(1), \ldots, f(p)$ are disjoint and each has measure $\frac{k+\varepsilon}{qk} > 1/q$. The construction of the sets $D_1, \ldots, D_m$ implies that $||C_i \cap f(j)|| = 1/p$ for every $i \in [n\binom{p}{q}]$ and $j \in [p]$.

The base of each of the rays forming the graph $P^n_{p,q,d}$ will be colored individually obeying that the set assigned to a vertex $(A, (d-1)/2)$ is a subset of $f(A)$. This guarantees that the fractional coloring of the subgraph of $P^n_{p,q,d}$ induced by the vertices of the bases is proper (recall that $f(j) \cap f(j') = \emptyset$ for $j \neq j'$).

Consider one of the rays $S^X_{p,q,(d-1)/2}$ forming the graph $P^n_{p,q,d}$ and let C_i be the set assigned to its special vertex; we modify the mapping f to a mapping $f' : [p] \hookrightarrow 2^{[0,p/q+\varepsilon)}$ such that $f'(j) \subseteq f(j)$ and $||f'(j)|| = 1/q$ for every $j \in [p]$. For $j \in X$, choose $f'(j)$ to be any subset of $f(j)$ such that $f'(j) \cap C_i = f(j) \cap C_i$ and $||f'(j)|| = 1/q$, i.e., $||f'(j) \cap C_i|| = 1/p$. Observe that $||f(X) \cap C_i|| = q/p = 1/k$. For $j \notin X$, choose $f'(j)$ to be any subset of $f(j)$ such that $||f'(j) \cap C_i|| = \max\{0, (1-\varepsilon)/p\}$ and $||f'(j)|| = 1/q$. The vertex $(A, (d-1)/2)$ of the base of $S^X_{p,q,(d-1)/2}$ is now assigned the set $f'(A)$. It follows that

$$||f'(X) \cap C_i|| = \frac{1}{k} \qquad \text{and} \qquad ||f'([p] \setminus X) \cap C_i|| = \max\left\{0, \frac{(k-1)(1-\varepsilon)}{k}\right\} .$$

If $(d-1)/2$ is even, the coloring can be extended to the ray by Lemma 7, and if $(d-1)/2$ is odd, it can be extended by Lemma 8. The proof is now completed. $\square$

4 Lower bounds

This section is devoted to proving that the bounds on ε given in Theorem 9 are the best possible for $k \in \{2\} \cup [3, \infty)$; the remaining case $k \in (2, 3)$ seems more difficult and we briefly discuss this case in Section 5.

Let us summarize the main ideas used in our lower bound arguments. The graphs that we use are universal graphs $P^n_{p,q,d}$. For simplicity, we focus on the case when $k + \varepsilon = p/q + \varepsilon$ is rational. First, cover the interval $[0, k+\varepsilon)$ with n unit sets in such a way that each point of the interval is contained in the same number of the sets. Each of the special vertices in the rays of the same type is then precolored with one of the n sets. By an averaging argument we show that any coloring of the base of $P^n_{p,q,d}$ behaves with respect to one of the rays at least as "bad" as the pseudorandom coloring used in the previous section. This will yield that our upper bounds are optimal.

Our exposition is started with two lemmas that are counterparts of Lemmas 7 and 8.

Lemma 10. *Let p, q and ℓ be integers, $1 \leq q \leq p$, $p/q \in \{2\} \cup [3,\infty)$, $\ell = 0 \bmod 2$ and $\ell \geq 2$, and $\varepsilon > 0$. Consider a fractional $(p/q+\varepsilon)$-precoloring of the special vertex and the vertices of the base of the graph $S_{p,q,\ell}$. Let C be the set assigned to the special vertex $([q],0)$ and C' the set assigned to the vertex $([q],\ell)$. If*

$$||C \cap C'|| < 1 - \frac{\ell\varepsilon}{2} \ , \tag{4}$$

then the precoloring cannot be extended to a fractional $(p/q+\varepsilon)$-coloring of $S_{p,q,\ell}$.

Proof. The proof proceeds by induction on ℓ; the base case of the induction is $\ell = 2$. Suppose that (4) holds and there exists a fractional $(p/q+\varepsilon)$-coloring of $S_{p,q,2}$. Let C_0 be the union of the sets assigned to the neighbors of the special vertex. Observe that every neighbor of the special vertex $([q],0)$ is also a neighbor of $([q],2)$. Hence, C_0 is disjoint both with C and C'.

If $k \geq 3$, the subgraph induced by the neighbors of the special vertex is isomorphic to $K_{p-q,q}$; in particular, its fractional chromatic number is $p/q-1$. Consequently, $||C_0|| \geq k-1$. If $k = 2$, it also holds that $||C_0|| \geq k-1 = 1$ since the neighborhood of the special vertex is non-empty and each vertex is assigned a set of measure at least one. Since the sets $C \cup C'$ and C_0 are disjoint, we obtain that $||C \cup C'|| \leq 1+\varepsilon$. The fact that $||C|| = ||C'|| = 1$ implies that $||C \cap C'|| \geq 1-\varepsilon$ which contradicts (4).

Suppose now that $\ell \geq 4$, (4) holds and there exists a fractional $(p/q+\varepsilon)$-coloring of the graph $S_{p,q,\ell}$ extending the given precoloring. Let C'' be the set assigned to the vertex $([q],\ell-2)$. Since the coloring of $S_{p,q,\ell}$ restricted to the special vertex and the vertices $(X,\ell-2)$, $X \in \binom{[p]}{q}$, can be viewed as a precoloring of the subgraph of $S_{p,q,\ell}$ induced by vertices (X,i) with $i \leq \ell-2$ and this precoloring can be extended to the whole graph, we obtain by induction that

$$||C \cap C''|| \geq 1 - \frac{(\ell-2)\varepsilon}{2} \ . \tag{5}$$

Similarly, the coloring of $S_{p,q,\ell}$ restricted to the subgraph induced by the vertex $([q], \ell-2)$ and the vertices (X, i) with $X \in \binom{[p]}{q}$ and $i \in \{\ell-1, \ell\}$ can be viewed as a precoloring of a subgraph isomorphic to $S_{p,q,2}$ and this precoloring can be extended to the whole subgraph induced by these vertices, the induction yields that

$$||C' \cap C''|| \geq 1 - \varepsilon \ . \tag{6}$$

Combining (5), (6) and $||C|| = ||C'|| = ||C''|| = 1$, we obtain that $||C \cap C'|| \geq 1 - \frac{\ell\varepsilon}{2}$ which violates (4). □

Lemma 11. *Let p, q and ℓ be integers, $1 \leq q \leq p$, $p/q \in \{2\} \cup [3, \infty)$, $\ell = 1 \mod 2$ and $\ell \geq 1$, and $\varepsilon > 0$. Consider a fractional $(p/q + \varepsilon)$-precoloring of the special vertex and the vertices of the base of the graph $S_{p,q,\ell}$. Let C be the set assigned to the special vertex $([q], 0)$ and C' the union of the sets assigned to the neighbors of the vertex $([q], \ell)$ in the base of $S_{p,q,\ell}$. If*

$$||C \cap C'|| > \frac{(\ell-1)\varepsilon}{2} \ , \tag{7}$$

then the precoloring cannot be extended to a fractional $(p/q+\varepsilon)$-coloring of $S_{p,q,\ell}$.

Proof. If $\ell = 1$, all the vertices of $S_{p,q,\ell}$ are colored and the coloring can be proper only if the sets C and C' are disjoint; hence, $||C \cap C'||$ must be equal to zero in this case.

Assume now that $\ell \geq 3$. Suppose that (7) holds and though there exists a fractional $(p/q+\varepsilon)$-coloring extending the precoloring. Consider this coloring restricted to the special vertex of $S_{p,q,\ell}$ and the vertices $(X, \ell-1)$, $X \in \binom{[p]}{q}$; this precoloring clearly extends to a subgraph of $S_{p,q,\ell}$ (isomorphic to $S_{p,q,\ell-1}$) induced by all its vertices except those forming its base. Lemma 10 implies that $||C \cap C''|| \geq 1 - \frac{(\ell-1)\varepsilon}{2}$ where C'' is the set assigned to the vertex $([q], \ell-1)$. Since the vertices $([q], \ell-1)$ and $([q], \ell)$ have the same neighbors in the base of $S_{p,q,\ell}$, it follows that $||C \cap C'|| \leq 1 - ||C \cap C''|| \leq \frac{(\ell-1)\varepsilon}{2}$ which contradicts (7). □

We are now ready to prove that the bounds on ε in Theorem 3 are the best possible if $k \in \{2\} \cup [3, \infty)$.

Theorem 12. *For every rational $k \in \{2\} \cup [3, \infty)$, every integer $d \geq 3$, and every $\varepsilon > 0$ not satisfying (1), there exists a graph G with fractional* *chromatic number k, a subset W of its vertices at mutual distance at least d and a fractional $(k+\varepsilon)$-precoloring of the vertices of W that cannot be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Proof. Let q be an integer such that kq is an integer; the exact value of q will be chosen (as a sufficiently large integer) later in the proof. Let $p = kq$ and let p' be the largest integer such that $p'/q \leq k+\varepsilon$.

Set G to be the graph $P^n_{p,q,d}$ where $n = \binom{p'}{q}$. Let $f : [p'] \hookrightarrow 2^{[0,k+\varepsilon)}$ be a function such that $f(i) = [(i-1)/q, i/q)$. Consider a precoloring of G assigning the n special vertices of the copies of $S^X_{p,q,\lfloor d/2 \rfloor}$ in $P^n_{p,q,d}$ all the n sets $f(X')$, $X' \in \binom{[p']}{q}$. We claim that this fractional $(k+\varepsilon)$-precoloring of G cannot be extended to the whole graph if q is sufficiently large. Since the precolored vertices are at mutual distance at least d, the statement of the theorem will then follow.

Suppose the opposite and consider the extension of the initial precoloring restricted to the special vertices of $P^n_{p,q,d}$ and the vertices in the bases of the rays forming the graph $P^n_{p,q,d}$. This new precoloring is what we further refer to as *the precoloring*.

Suppose first that $d = 0 \bmod 4$. Let C be the set assigned to the vertex obtained by identifying the vertices $([q], d/2)$. in the bases of the rays. A double counting argument yields the following:

$$||C \cap [0, p'/q)|| = \frac{1}{\binom{p'-1}{q-1}} \sum_{X \in \binom{p'}{q}} ||C \cap f(X)|| \ .$$

Indeed, each point of $[0, p'/q)$ is contained in $\binom{p'-1}{q-1}$ sets $f(X)$ and thus the equality follows. Since $||C \cap [0, p'/q)|| \leq 1$, there exists $X \in \binom{[p']}{q}$ such that

$$||C \cap f(X)|| \leq \frac{\binom{p'-1}{q-1}}{\binom{p'}{q}} = \frac{q}{p'} \ .$$

As q grows to the infinity, $\frac{q}{p'}$ tends to $1/(k+\varepsilon)$. Hence, for q sufficiently large, it follows that $q/p' < 1 - d\varepsilon/4$. Lemma 10 then excludes the existence of an extension of the precoloring to a fractional $(k+\varepsilon)$-coloring in the copy of $S_{p,q,d/2}$ with the special vertex precolored with $f(X)$.

Suppose now that $d = 2 \bmod 4$. Let C be the union of the sets assigned to the vertices obtained by identifying the vertices in the bases of the rays that are the neighbors of the vertex $([q], d/2)$. Since the fractional chromatic number of the subgraph induced by these neighbors is $k-1$, we obtain by a double counting argument the following:

$$k-1-(k+\varepsilon-p'/q) \leq ||C \cap [0, p'/q)|| = \frac{1}{\binom{p'-1}{q-1}} \sum_{X \in \binom{p'}{q}} ||C \cap f(X)|| \; .$$

Hence, there exists $X \in \binom{[p']}{q}$ such that

$$||C \cap f(X)|| \geq \frac{k-1-(k+\varepsilon-p'/q)}{p'/q} \geq \frac{k-1-(k+\varepsilon-p'/q)}{k+\varepsilon} \; .$$

As q grows to the infinity, $k+\varepsilon-p'/q$ tends to 0. Hence, for q sufficiently large, there exists $\varepsilon' > \varepsilon$ such that ε' does not satisfy (1) and $||C \cap f(X)|| \geq (k-1)/(k+\varepsilon')$. Since $(k-1)/(k+\varepsilon') > (d-2)\varepsilon'/4$ as ε' violates (1), there is no extension of the precoloring to a fractional $(k+\varepsilon')$-coloring by Lemma 11 in the copy of $S_{p,q,d/2}$ with the special vertex precolored with $f(X)$; in particular, there is no extension to a fractional $(k+\varepsilon)$-coloring.

In the remainder of the proof, we deal with the case when d is odd. Let C_X, $X \in \binom{[p]}{q}$, be the union of all the sets assigned to the vertices $(X, (d-1)/2)$ of the bases of the rays that form the graph $P^n_{p,q,d}$. Observe that the sets C_X and $C_{X'}$ are disjoint whenever $X \cap X' = \emptyset$. Hence, each point of the interval $[0, k+\varepsilon)$ is contained in at most $\binom{p-1}{q-1}$ of these sets by Erdős-Ko-Rado theorem (see, e.g., [10]).

We now suppose that $d = 1 \bmod 4$. Since each point of the interval $[0, k+\varepsilon)$ is contained in at most $\binom{p-1}{q-1}$ of the sets C_X, $X \in \binom{[p]}{q}$, we obtain that

$$\sum_{X \in \binom{[p]}{q}} ||C_X \cap [0, p'/q)|| \leq \binom{p-1}{q-1} \frac{p'}{q} \; .$$

Consequently, there exists $X \in \binom{[p]}{q}$ such that $||C_X \cap [0, p'/q)|| \leq \frac{p'}{kq}$. A double counting argument analogous to that used in the case $d = 0 \bmod 4$ yields that there exists $X' \in \binom{[p']}{q}$ such that

$$||C_X \cap f(X')|| \leq \frac{p'}{kq} \cdot \frac{\binom{p'-1}{q-1}}{\binom{p'}{q}} = 1/k \; .$$

Consider now the copy of $S^X_{p,q,(d-1)/2}$ such that its special vertex is precolored with $f(X')$. The measure of the intersection of the set assigned to the vertex $(X,(d-1)/2)$ of the base, which is a subset of C_X, and the set $f(X')$ is at most $||C_X \cap f(X')|| \le 1/k$. Since ε does not satisfy (1), it holds that $1/k < 1-(d-1)\varepsilon/4$. By Lemma 10, the precoloring of the special vertex and the base cannot be extended to the whole ray.

The last case to be considered is $d = 3 \bmod 4$. We claim that there exists $Y \in \binom{[p]}{q}$ such that

$$\sum_{X\in\binom{[p]\setminus Y}{q}} ||C_X \setminus [0,1)|| \le (k+\varepsilon-1)\cdot\frac{\binom{p-1}{q-1}\binom{p-q}{q}}{\binom{p}{q}} = \frac{k+\varepsilon-1}{k}\binom{p-q}{q}.$$

Indeed, each point of the interval $[1,k+\varepsilon)$ is contained in at most $\binom{p-1}{q-1}$ of the sets C_X, $X \in \binom{[p]}{q}$, and each set X is contained in the sets $\binom{[p]\setminus Y}{q}$ for $\binom{p-q}{q}$ choices of Y, so the claim follows. By symmetry, we can (and we will) assume that $Y = [q]$.

Consider now the copy of the ray $S^{[q]}_{p,q,(d-1)/2}$ in $P_{p,q,d}$ with the special vertex precolored with $f([q]) = [0,1)$. Let C^0_X be the set assigned to the vertex $(X,(d-1)/2)$ of this copy. Since $C^0_X \subseteq C_X$, we obtain that

$$\sum_{X\in\binom{[p]\setminus[q]}{q}} ||C^0_X \cap [0,1)|| \ge \frac{1-\varepsilon}{k}\binom{p-q}{q}.$$

Since each point of $[0,1)$ can be contained in at most $\binom{p-q-1}{q-1}$ of the sets C^0_X, $X \in \binom{[p]\setminus[q]}{q}$, it follows that

$$||[0,1) \cap \bigcup_{X\in\binom{[p]\setminus[q]}{q}} C^0_X|| \ge \frac{\frac{1-\varepsilon}{k}\binom{p-q}{q}}{\binom{p-q-1}{q-1}} = (1-\varepsilon)\left(1-\frac{q}{p}\right). \tag{8}$$

Since ε violates (1), the last expression in (8) is greater than $\frac{d-3}{4}\varepsilon$. Consequently, the precoloring of the special vertex and the base of $S^X_{p,q,(d-1)/2}$ cannot be extended to the whole ray by Lemma 11. □

5 Conclusion

Theorem 3 gives the best possible condition on ε for $k \in \{2\}\cup[3,\infty)$ and any $d \ge 3$. One would be tempted to conjecture that the condition (1) on ε is

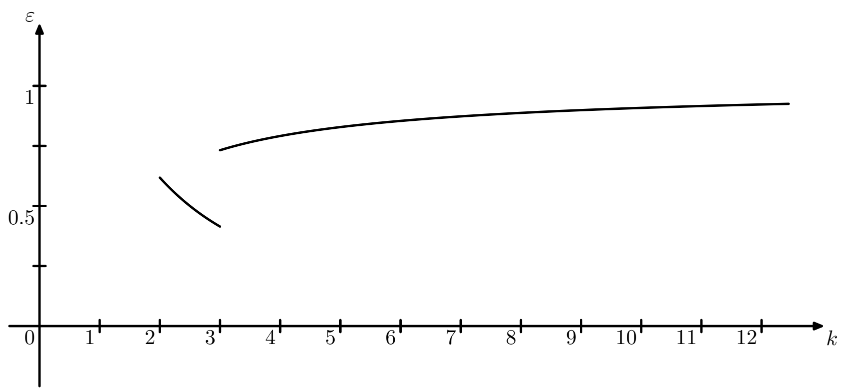

Figure 4: The dependence of the minimum value of ε on k for $d = 4$.

also the best possible for $k \in (2, 3)$. However, this is not the case. For $d = 4$, three of the authors together with Jan van den Heuvel and Jean-Sébastien Sereni were able to determine optimal minimum values of ε for $k \in (2, 3)$ (see Figure 4): a fractional $(k + \varepsilon)$-precoloring of vertices at distance at least four of any graph G with fractional chromatic number $k \in (2, 3)$ can be extended to a fractional $(k + \varepsilon)$-coloring of G if $\varepsilon^2 + (k - 1)\varepsilon - 1 \geq 0$. This condition on ε is also the best possible in the sense of Theorem 3. It has suprised us that ε as a function of k for $d = 4$ is not continous. We intend to investigate this phenomenom further.

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