d-representability of simplicial complexes of fixed dimension *

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Abstract

Let K be a simplicial complex with vertex set $V = \{v_1, \ldots, v_n\}$. The complex K is d-representable if there is a collection $\{C_1, \ldots, C_n\}$ of convex sets in $\mathbb{R}^d$ such that a subcollection $\{C_{i_1}, \ldots, C_{i_j}\}$ has a nonempty intersection if and only if $\{v_{i_1}, \ldots, v_{i_j}\}$ is a face of K.

In 1967 Wegner proved that every simplicial complex of dimension d is $(2d+1)$-representable. He also conjectured that his bound is the best possible, it is, that there are d-dimensional simplicial complexes which are not $2d$-representable.

We prove that his conjecture was indeed right. Thus we add another piece to the puzzle of intersection patterns of convex sets in Euclidean space.

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1 Introduction

Let $\mathcal{C}$ be a collection of sets. The *nerve* of $\mathcal{C}$ is a simplicial complex[1] with vertex set $\mathcal{C}$ and with faces of the form $\{C_1, \cdots, C_k\} \subseteq \mathcal{C}$ such that the intersection $C_1 \cap \cdots \cap C_k$ is nonempty. We say that a simplicial complex is *d-representable* if it is isomorphic to the nerve of a finite collection of convex sets in $\mathbb{R}^d$. This notion is designed to capture possible 'intersection patterns' of convex sets in $\mathbb{R}^d$. The study of intersection patterns of convex sets dates back to a theorem of Helly [Hel23].

Let us also mention that d-representable simplicial complexes are very closely related to well studied intersection graphs of convex sets. An intersection graph only records which pairs of convex sets have a nonempty intersection; however, it does not take care of multiple intersections. Thus d-representable complexes provide more detailed information about the intersection pattern.

The need to understand intersection patterns of convex sets appears for instance in manifold learning. The task might be to reconstruct the homotopy type of a manifold M given by sample points $\{p_i\}$. Sample points can be enlarged to convex sets $\{C_i\}$; and under certain conditions (which we do not discuss here) M is homotopic to $\bigcup C_i$. On the other hand, via the nerve theorem, $\bigcup C_i$ is homotopic to the nerve of $\{C_i\}$. See, for example, [AL10] and the references therein for more details. One of the pioneering results in this area was an algorithm for reconstruction α-shapes by Edelsbrunner and Mücke [EM94].

The reader is referred to [Eck93] or [Tan11] for more background on intersection patterns of convex sets.

One question arising in this area is how the dimension of a complex affects d-representability. Wegner [Weg67] showed that a complex of dimension d is always $(2d+1)$-representable. (This result was also independently found by Perel'man [Per85].) Wegner also conjectured that the value $2d+1$ is the best possible, i.e., that there are

[1] We assume that the reader is familiar with simplicial complexes; otherwise we refer to standard sources such as [Hat01, Mun84, Mat03].

d-dimensional simplicial complexes which are not $2d$-representable. (The question about the best possible value is also reproduced by Eckhoff [Eck93], and the author is not aware that this question has been answered yet.)

Wegner proved that the barycentric subdivision[2] of a nonplanar graph is not 2-representable. He also conjectured that the barycentric subdivision of a d-dimensional complex that does not embed into $\mathbb{R}^{2d}$ is not $2d$-representable; however, he was not able to prove his conjecture.

In this short note we prove that the value $2d+1$ is indeed the best possible. Let Δ_n denote the n-dimensional simplex (as an abstract simplicial complex) and let $\mathsf{K}^{(k)}$ denote the k-skeleton of a simplicial complex K. We prove that the barycentric subdivision of $\Delta_{2d+2}^{(d)}$ and the barycentric subdivisions of many other complexes are not d-representable; see the precise statement below.

Theorem 1.1. *The barycentric subdivision of $\Delta_{2d+2}^{(d)}$ is not $2d$-representable. More generally, if L is a d-dimensional simplicial complex with non-vanishing Van Kampen obstruction, then the barycentric subdivision* $\operatorname{sd}\mathsf{L}$ *is not $2d$-representable.*

Remark 1.2. *Van Kampen obstruction* is a certain cohomology obstruction for the embeddability d-dimensional simplicial complexes into $\mathbb{R}^{2d}$. We are not going to define this obstruction precisely since we would need too many preliminaries. The interested reader is referred either to [Mel09] for a survey or to [MTW11, Appendix D] for an elementary exposition.

Let us just mention some properties of Van Kampen obstruction. If K is a d-dimensional simplicial complex which embeds into $\mathbb{R}^{2d}$, then its Van Kampen obstruction has to vanish. If $d \neq 2$, then also the converse is true, it is, a d-dimensional simplicial complex with vanishing Van Kampen obstruction embeds into $\mathbb{R}^{2d}$. In case $d = 2$

[2] In this case, every edge is subdivided into two edges and a new vertex in the center of the edge is inserted.

there are, however, simplicial complexes with vanishing Van Kampen obstruction which do not embed into $\mathbb{R}^4$; see [FKT94].

Regarding our proof method, let us first indicate Wegner's approach for case $d = 1$. Given an graph G (graph is a 1-dimensional simplicial complex) whose barycentric subdivision is 2-representable, Wegner constructs a piecewise linear embedding of $|\operatorname{sd} G|$ into $\mathbb{R}^2$, thereby proving that G must be planar.

It seems hard to extend this construction in such a way that g would be an embedding in higher dimensions. Our main observation is that it is not necessary to require that g is an embedding in order to obtain a contradiction with an embeddability-type result. We only construct such a g that disjoint simplices have disjoint images, which is still in contradiction with non-vanishing Van Kampen obstruction.

2 Barycentric subdivision

In order to set up notation, we recall the definition of a barycentric subdivision of a simplicial complex.

From geometric point of view we put a new vertex into the barycenter of every geometric face of a simplicial complex K. Then we form a new simplicial complex whose vertices are the barycenters and whose faces are simplices formed between these barycenters.

It is perhaps more convenient to state the precise definition in abstract setting. Given a simplicial complex K the *barycentric subdivision* of K is a simplicial complex $\operatorname{sd}\mathsf{K}$ whose set of vertices is the set $\mathsf{K} \setminus \emptyset$ and whose faces are collections $\{\alpha_1, \dots, \alpha_m\}$ of faces of K such that

$$\alpha_1 \supsetneq \alpha_2 \supsetneq \cdots \supsetneq \alpha_m \neq \emptyset.$$

See Figure 1.

The complexes K and $\operatorname{sd}\mathsf{K}$ have the same geometric realizations: $|\mathsf{K}| = |\operatorname{sd}\mathsf{K}|$.

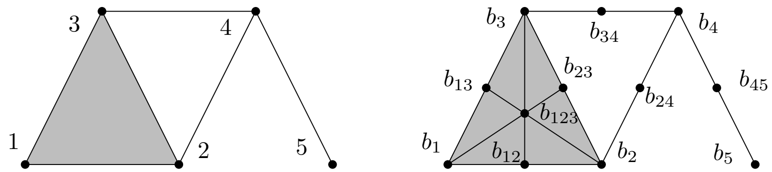

Figure 1: Barycentric subdivision of a complex. For example, the vertex b_{13} denotes the barycenter of the face $13 = \{1, 3\}$ (in geometric setting).

3 Proof

For the proof we will need two auxiliary results.

Theorem 3.1 (Van Kampen - Flores theorem; see [vK32, Flo34], [Mat03, Theorem 5.1.1]). *Let* $\mathsf{K} = \Delta^{(d)}_{2d+2}$. *Then for any continuous map* $f\colon |\mathsf{K}| \to \mathbb{R}^{2d}$ *there are two disjoint* d*-dimensional simplices* γ *and* δ *of* K *whose images* $f(|\gamma|)$ *and* $f(|\delta|)$ *intersect.*

In case $d = 1$ the Theorem 3.1 implies that the graph K_5 is not planar; however, the theorem is actually even stronger: In any drawing of K_5 into the plane there are two non-adjacent edges which intersect in the drawing.

We remark that the conclusion of the theorem remains true if K is replaced with any d-dimensional complex with non-zero Van Kampen obstruction (in particular, K has a non-zero Van Kampen obstruction). The fact that Theorem 3.1 extends to complexes with non-zero obstruction just follows from one of possible definitions of Van Kampen obstruction (and is trivial for a reader familiar with this topic); see exposition in [FKT94].[3] On the other hand, Theorem 3.1 for our specific K can be proved on more elementary level

[3] There is a sign error in [FKT94] in the definition of Van Kampen obstruction observed by Melikhov [Mel09]. However, it does not affect our conclusion.

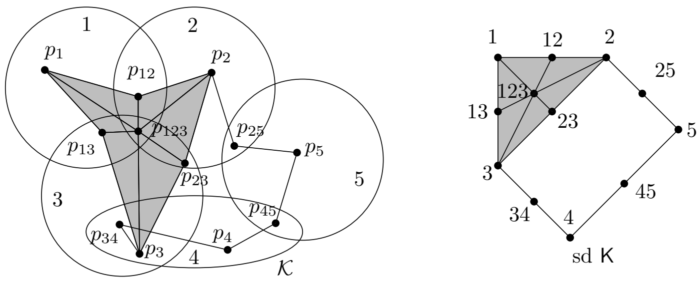

Figure 2: Mapping $\operatorname{sd}\mathsf{K}$ into $\mathcal{K}$. The notation is simplified. For instance 12 stands for $\{1,2\}$, p_{123} stands for $p(\{1,2,3\})$, etc.

using the Borsuk-Ulam theorem; and that is why we also emphasize this specific case.

Let α and β be faces of a simplicial complex K. We say that α and β are *remote* if there is no edge $ab \in \mathsf{K}$ with $a \in \alpha, b \in \beta$.

Lemma 3.2. *Let $\mathcal{K}$ be a collection of convex sets in $\mathbb{R}^m$ and let $\mathsf{K} := \mathsf{N}(\mathcal{K})$ be the nerve of $\mathcal{K}$. Then there is a map $g : |\operatorname{sd}\mathsf{K}| \to \mathbb{R}^m$, linear on every simplex, such that $g(|\operatorname{sd}\alpha|) \cap g(|\operatorname{sd}\beta|) = \emptyset$ for any remote $\alpha, \beta \in \mathsf{K}$.*

Proof. First we specify g on the vertices of $\operatorname{sd}\mathsf{K}$ then we extend it linearly to each simplex of $\operatorname{sd}\mathsf{K}$. See Figure 2.

A vertex of $\operatorname{sd}\mathsf{K}$ is a simplex of K, it is, a subcollection $\mathcal{K}'$ of $\mathcal{K}$ with a nonempty intersection. Let us pick a point $p(\mathcal{K}')$ in $\cap\mathcal{K}'$. We set $g(\mathcal{K}') := p(\mathcal{K}')$ for each $\mathcal{K}' \in \mathsf{K}$. As we already mentioned, we extend g linearly to $\operatorname{sd}\mathsf{K}$.

If $\alpha = \mathcal{K}' \in \mathsf{K}$, then $g(|\operatorname{sd}\alpha|) \subseteq \cup\mathcal{K}'$. Thus $g(|\operatorname{sd}\alpha|) \cap g(|\operatorname{sd}\beta|) = \emptyset$ for remote $\alpha, \beta \in \mathsf{K}$.

□

Proof of Theorem 1.1. First we prove the specific case.

Let $\mathsf{K} = \operatorname{sd} \Delta^{(d)}_{2d+2}$. For contradiction we assume that K is $2d$-representable. Let $\mathcal{K}$ be a $2d$-representation of it. (Without loss of generality $\mathsf{K} = \mathsf{N}(\mathcal{K})$.) According to Lemma 3.2 there is a map $g\colon |\operatorname{sd} \mathsf{K}| \to \mathbb{R}^{2d}$ such that $g(|\operatorname{sd} \alpha|) \cap g(|\operatorname{sd} \beta|) = \emptyset$ for any remote $\alpha, \beta \in \mathsf{K}$.

Since $\operatorname{sd} \mathsf{K} = \operatorname{sd} \operatorname{sd} \Delta^{(d)}_{2d+2}$, we have $|\Delta^{(d)}_{2d+2}| = |\mathsf{K}| = |\operatorname{sd} \mathsf{K}|$, and thus we can also think of g as a piecewise linear map from $\Delta^{(d)}_{2d+2}$ to $\mathbb{R}^{2d}$.

Let γ and δ be disjoint simplices of $\Delta^{(d)}_{2d+2}$. Let α be a simplex of $\operatorname{sd} \gamma$ and β a simplex of $\operatorname{sd} \delta$. Then α and β are remote in K. Thus $g(|\operatorname{sd} \alpha|) \cap g(|\operatorname{sd} \beta|) = \emptyset$. Consequently, $g(|\gamma|) \cap g(|\delta|) = \emptyset$ for any choice of disjoint γ and δ. However, this contradicts the Van Kampen-Flores theorem.

More general part of the theorem is obtained along the same lines when a generalized version of Theorem 3.1 is used. $\square$

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References

[AL10] D. Attali and A. Lieutier. Reconstructing shapes with guarantees by unions of convex sets [extended abstract]. In *Computational geometry (SCG'10)*, pages 344–353. ACM, New York, 2010.

[Eck93] J. Eckhoff. Helly, Radon and Carathéodory type theorems. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry.* North-Holland, Amsterdam, 1993.

[EM94] H. Edelsbrunner and E. P. Mücke. Three-dimensional alpha shapes. *ACM Trans. Graph.*, 13:43–72, 1994.

[FKT94] M. H. Freedman, V. S. Krushkal, and P. Teichner. Van Kampen's embedding obstruction is incomplete for 2-complexes in $\mathbb{R}^4$. *Math. Res. Lett.*, 1(2):167–176, 1994.

[Flo34] A. Flores. Über n-dimensionale Komplexe die im R_{2n+1} absolut selbstverschlungen sind. *Ergeb. Math. Kolloq.*, 4:6–7, 1932/1934.

[Hat01] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2001.

[Hel23] E. Helly. Über mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Deustch. Math.-Verein.*, 32:175–176, 1923.

[Mat03] J. Matoušek. *Using the Borsuk-Ulam Theorem.* Springer, Berlin etc., 2003.

[Mel09] S. A. Melikhov. The van Kampen obstruction and its relatives. *Tr. Mat. Inst. Steklova*, 266(Geometriya, Topologiya i Matematicheskaya Fizika. II):149–183, 2009.

[MTW11] J. Matoušek, M. Tancer, and U. Wagner. Hardness of embedding simplicial complexes in $\mathbb{R}^d$. *J. Eur. Math. Soc. (JEMS)*, 13(2):259–295, 2011.

[Mun84] J. R. Munkres. *Elements of Algebraic Topology.* Addison - Wesley, 1984.

[Per85] G. Ya. Perel'man. Realization of abstract k-skeletons of intersections of convex polyhedra in R^{2k-1}. *Geometric Questions in the Theory of Functions and Sets (Kalinin. Gos. Univ., Kalinin)*, pages 129–131, 1985. In Russian.

[Tan11] M. Tancer. Intersection patterns of convex sets via simplicial complexes, a survey. Preprint; http://arxiv.org/abs/1102.0417, 2011.

[vK32] R. E. van Kampen. Komplexe in euklidischen Räumen. *Abh. Math. Sem. Hamburg*, 9:72–78, 1932. Berichtigung dazu, *ibid.* (1932) 152–153.

[Weg67] G. Wegner. *Eigenschaften der Nerven homologisch-einfacher Familien im* R^n. PhD thesis, Universität Göttingen, 1967. In German.