# Homomorphism Bounds for Oriented Planar Graphs of Given Minimum Girth

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**Abstract**

We find necessary conditions for a digraph $H$ to admit a homomorphism from every oriented planar graph of girth at least $n$, and use these to prove the existence of a planar graph of girth 6 and oriented chromatic number at least 7. We identify a $K_4$-free digraph of order 7 which admits a homomorphism from every oriented planar graph (here $K_n$ means a digraph with $n$ vertices and arcs in both directions between every distinct pair), and a $K_3$-free digraph of order 4 which admits a homomorphism from every oriented planar graph of girth at least 5.

## 1 Introduction

This paper is concerned with homomorphisms of digraphs. Unless stated otherwise, all digraphs will be finite and have no have loops or multiple arcs in the same direction between the same pair of vertices. In other words digraphs are irreflexive relations on their vertex sets. If $G$ is a digraph, then we write $G = (V, A)$, where $V = V(G)$ and $A = A(G)$ are respectively the vertex set and arc set of $G$. An

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*oriented* graph is a digraph without opposite arcs, and an *unoriented* graph is a symmetric digraph. Unoriented graphs are usually defined in terms of undirected edges, and we can identify these in an obvious way with pairs of opposite arcs. Here we are regarding unoriented graphs as a particular type of digraph.

If $G$ is an unoriented graph, then an *orientation* of $G$ is an oriented graph obtained by replacing each edge $\{u, v\}$ by one of the arcs $(u, v)$ or $(v, u)$ (or, in terms of our original definition, by deleting one arc from each opposite pair). If $\mathcal{K}$ is a class of unoriented graphs, then $\overrightarrow{\mathcal{K}}$ will denote the class of orientations of graphs in $\mathcal{K}$.

A *homomorphism* (resp. *anti-homomorphism*) from digraph $G_1 = (V_1, A_1)$ to digraph $G_2 = (V_2, A_2)$ is a function $\phi : V_1 \to V_2$ such that, if $(u, v) \in A_1$, then $(\phi(u), \phi(v)) \in A_2$ (resp. $(\phi(v), \phi(u)) \in A_2$). A digraph $H$ is a *homomorphism bound* for a class $\mathcal{C}$ of digraphs–in short a $\mathcal{C}$*-bound*–if every digraph in $\mathcal{C}$ admits a homomorphism to $H$.

Given a class $\mathcal{C}$ of digraphs, a natural first question is: does there exist a $\mathcal{C}$-bound, and if so, what properties must such a bound have? When $\mathcal{C}$ is a class of unoriented graphs, then clearly we may as well confine our attention to the case where the target graph is also unoriented. An unoriented graph is $n$-colorable if and only if it admits a homomorphism to $K_n$. For this reason, we may refer to any homomrphism $G \to H$ as an $H$*-colouring*, and the vertices of $H$ as *colours*. We can express many well known theorems in terms of homomorphism bounds. For example the 4CT says that $K_4$ bounds the class of unoriented planar graphs (and so also the class of planar digraphs).

Homomorphism bounds have been also been intensively studied in the case where $\mathcal{C}$ is a class of oriented graphs, and the homomorphism bound is also required to be oriented. Particular attention has focused on the problem of how small such bounds can be made. By analogy with the unoriented case, we define the *oriented chromatic number* $\chi_o(G)$ of an oriented graph $G$, to be the smallest order of an oriented graph admitting a homomorphism from $G$ (that is of an oriented homomorphism bound for $\{G\}$). If $G$ is an unoriented graph, then we define $\chi_o(G)$ to be the maximum value of $\chi_o(\overrightarrow{G})$ taken over all orientations $\overrightarrow{G}$ of $G$. For a class $\mathcal{C}$ of unoriented graphs we define $\chi_o(\mathcal{C}) = \sup_{G \in \mathcal{C}} \chi_o(G)$. Clearly $\chi_o(\mathcal{C})$ is bounded above by the order of the smallest $\overrightarrow{\mathcal{C}}$-bound (if such exists), with equality when $\mathcal{C}$ is complete, that is when the disjoint union of two graphs in $\mathcal{C}$ is a subgraph of a graph in $\mathcal{C}$ (an easy exercise: see [7] for more details).

In this paper we are mainly concerned with homomorphism bounds for $\overrightarrow{\mathcal{P}_n}$, where $\mathcal{P}_n$ is the class of planar graphs of girth at least $n$. Since $\mathcal{P}_n$ is clearly complete, the smallest order of an oriented $\overrightarrow{\mathcal{P}_n}$-bound is the same as $\chi_o(\mathcal{P}_n)$. The problem of determining $\chi_o(\mathcal{P})$, where $\mathcal{P}$ is the class of all planar graphs, is analogous to the 4 colour problem. The smallest known $\overrightarrow{\mathcal{P}}$-bound has 80 vertices. That is $\chi_o(\mathcal{P}) \leq 80$ [15]. In the other direction it is known that $\chi_o(\mathcal{P}) \geq 18$ [8]. This leaves a very wide gap, which seems difficult to narrow in either direction. Better bounds are known for $\chi_o(\mathcal{P}_n)$, with the exact values being known for $n \geq 12$. The best known bounds are listed in the following (the reference for lower bound is given first in each case).

**Theorem 1** *[3, 4, 5, 8, 9, 10, 11, 12, 13, 15]*

1. $18 \leq \chi_o(\mathcal{P}) \leq 80$ *[8, 15]*
2. $11 \leq \chi_o(\mathcal{P}_4) \leq 40$ *[11, 12]*
3. $6 \leq \chi_o(\mathcal{P}_5) \leq 16$ *[10, 13]*
4. $6 \leq \chi_o(\mathcal{P}_6) \leq 11$ *[10, 5]*
5. $6 \leq \chi_o(\mathcal{P}_7) \leq 7$ *[10, 3]*
6. $5 \leq \chi_o(\mathcal{P}_8) \leq 7$ *[10, 3]*
7. $5 \leq \chi_o(\mathcal{P}_n) \leq 6$ *for* $9 \leq n \leq 11$ *[10, 9]*
8. $\chi_o(\mathcal{P}_n) = 5$ *for* $n \geq 12$ *[10, 4]*

In Section 3 we find some necessary conditions for a digraph to be a $\mathcal{P}_n$-bound. These give the new lower bound

**Theorem 2** $\chi_o(\mathcal{P}_6) \geq 7$

The proof of this result leads naturally to questions about $\overrightarrow{\mathcal{P}_n}$-bounds which are not necessarily oriented graphs. Clearly a digraph $H$ will bound $\overrightarrow{\mathcal{P}}$ if it contains an (unoriented) $K_4$, and $\overrightarrow{\mathcal{P}_4}$ if it contains an (unoriented) $K_3$, by the four colour theorem and Grötzsch's theorem respectively. An interesting (and still open) problem is how small can we make bounds which do *not* contain these graphs. We exhibit a $K_4$-free $\overrightarrow{\mathcal{P}}$-bound of order 7, and a $K_3$-free digraph $\overrightarrow{\mathcal{P}_5}$-bound of order 4. The latter is the smallest possible, but it is possible that there may be a $K_4$-free $\overrightarrow{\mathcal{P}}$-bound with as few as 5 vertices. If so we have yet another generalization of the 5-colour theorem.

We prove these results in Section 4 by adapting Thomassen's celebrated proof of the 5-choosability of planar graphs [17], and his similar proof of 3-choosability of graphs in $\mathcal{P}_5$ [20]. The changes are simple, and once they are made, the original arguments apply *mutatis mutandis*. For this reason we omit details of the proofs.

Finally we remark that all our definitions make sense even if digraphs are permitted to have loops. We exclude these just for convenience: if $H$ has a loop at $v$ then *every* digraph admits a homomorphism to $H$ (namely the constant map $v$), so disallowing loops excludes only trivial results.

# 2 Preliminaries

We introduce some more definitions and notations. A set with $k$ elements is called a *$k$-set*. The *complement* of a digraph $G$, written $\overline{G}$, is the digraph with the same vertex set as $G$, with $(u, v) \in \overline{G}$ if and only if $(u, v) \notin G$ $(u \neq v)$. An (anti-) isomorphism of a digraph $G$ is a bijective (anti-)homomorphism from $G$ to itself. The following observation is useful.

**Lemma 3** *If $\mathcal{C}$ is a class of unoriented graphs, $H$ is a $\overrightarrow{\mathcal{C}}$-bound and $H'$ is anti-isomorphic to $H$, then $H'$ is also a $\overrightarrow{\mathcal{C}}$-bound.*

**Proof:-** Let $G \in \overrightarrow{\mathcal{C}}$, then the oriented graph $G'$ obtained by reversing the arcs of $G$ is also in $\overrightarrow{\mathcal{C}}$, and so admits a homomorphism to $H$. The same map (of vertices) is an anti-homomorphism $G \to H$, which composed with the anti-isomorphism $H \to H'$ gives a homomorphism $G \to H'$. $\square$

Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots \epsilon_k) \in \{-1, 1\}^k$. We set $|\boldsymbol{\epsilon}| = k$. We define an $\boldsymbol{\epsilon}$*-path (trail)* from $v$ to $w$ to be a path (trail) $v = v_0, v_1, \ldots, v_k = w$ such that there is an arc from $v_{i-1}$ to $v_i$ if $\epsilon_i = 1$, and from $v_i$ to $v_{i-1}$ if $\epsilon_i = -1$.

**Definition:-** Given a digraph $G$, $v \in V(G)$ and $\boldsymbol{\epsilon} \in \{-1, 1\}^k$

$$N^{\boldsymbol{\epsilon}}(v) = \{w \in V(G) \mid \text{ there is an } \epsilon\text{-trail from } v \text{ to } w\}$$

We also write $out(v) = N^{(1)}(v)$, $in(v) = N^{(-1)}(v)$, $d^+(v) = |out(v)|$ and $d^-(v) = |in(v)|$.

**Definition:-** If $\Gamma$ is a digraph, the *nth power* of $\Gamma$, $\Gamma^n$ has the same vertex set as $\Gamma$, with an arc from $v$ to $w$ in $\Gamma^n$ if there is a $(1, 1, 1, \ldots 1)$-trail of length $n$ from $v$ to $w$. Note that, if $\Gamma$ is an oriented graph, then $\Gamma^k$ will in general have opposite edges and loops, but that $\Gamma^2$ has no loops.

**Definition:-** A class $\mathcal{C}$ of oriented graphs is *$k$-complete* if the graph obtained by pasting together two graphs in $\mathcal{C}$ along isomorphic $l$-tournaments ($l \leq k$) is also in $\mathcal{C}$. In particular, observe that each $\overrightarrow{\mathcal{P}_n}$ is 2-complete.

We say that an oriented graph $\Gamma$ is *minimal* with some property $P$ if $\Gamma$ has this property, but no proper subgraph of $\Gamma$ has.

**Lemma 4 ([16], Lemma 3)** *Let $\mathcal{C}$ be a 2-complete class of oriented graphs, $\Gamma$ a minimal $\mathcal{C}$-bound, $G$ a graph in $\mathcal{C}$, $(u, v)$ an arc in $G$ and $(u', v')$ an arc in $\Gamma$, then there is a homomorphism $\phi : G \to \Gamma$, such that $\phi(u) = u'$ and $\phi(v) = v'$.*

**Remark:-** This result was originally stated for $\Gamma$ an oriented graph, but applies (with identical proof) when $\Gamma$ is any digraph.

Let $q \equiv 3 \bmod 4$ be prime. The *Paley Tournament* $P_q$ is defined to be the tournament with vertex set $\mathbb{Z}_q$ and an arc from $v$ to $w$ if $w - v$ is a non-zero quadratic residue modulo $q$. (Since $q \equiv 3 \bmod 4$, -1 is not a quadratic residue, and so there are no opposite edges.) We then define $P_q^*$ to be the subgraph of $P_q$ induced by $\mathbb{Z}_q^* := \mathbb{Z}_q \setminus \{0\}$. When $k \in \mathbb{Z}_q^*$, the map $v \to kv$ is an isomorphism of $P_q^*$ when $k$ is a quadratic residue, and an anti-isomorphism when it is not. In particular, the tournament $P_7^*$ has vertex set $\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$. The three quadratic residues (1,2 and 4) have outdegree 3 and indegree 2, while the non-quadratic residues (3,5 and 6) have indegree 3 and outdegree 2. The group of isomorphisms acts transitively on the quadratic residues, and the group of isomorphisms and anti-isomorphisms acts transitively on the whole vertex set.

We finish this section by listing the tournaments of order at most 6 for which each vertex has indegree and outdegree at least 2. Up to isomorphism there are 6 of these tournaments: $T_0$=the circulant $C(5; 1, 2)$, $T_1$=the circulant $C(6; 1, 2)$ with vertex set $\mathbb{Z}_6$ and added arcs $(0, 3)$, $(1, 4)$ and $(2, 5)$; $T_2 = T_1$ with the arc $(1, 4)$ reversed; $T_3 = P_7^*$, $T_4 = T_3$ with the arc $(1, 3)$ reversed, and $T_5 = T_3$ with the arc $(1, 5)$ reversed. $T_5$ is anti-isomorphic to $T_4$ (via the map $x \to 5x$), and the rest are self anti-isomorphic.

**Lemma 5** *Every proper subgraph of $P_7^*$ either has a vertex with indegree or outdegree at most 1 or is isomorphic to a subgraph of $T_4$ or $T_5$.*

**Proof:-** Let $H$ be a proper subgraph of $P_7^*$. Suppose that $(u, v)$ is an arc missing from $H$. If $|H| \leq 5$, $v$ is a quadratic residue or $u$ is not, then $H$ has a vertex with indegree or outdegree at most 1. In the remaining case $|H| = 6$, $u$ is a quadratic residue and $v$ is not. By symmetry, we may assume $u = 1$, whence $v \in \{3, 5\}$, so that $H$ is isomorphic to a subgraph of $T_4$ or $T_5$. $\square$

## 3 Bounds for planar graphs

Let $\mathcal{O}$ and $\mathcal{O}_g$ denote respectively the classes of outerplanar graphs, and of outerplanar graphs of girth at least $g$. Pinlou and Sopena [14] have proved

**Theorem 6** $\chi_o(\mathcal{O}_g) = 7$ *for* $g = 3$, $6$ *for* $g = 4$ *and* $5$ *for* $g \geq 5$

The next result similarly gives the minimum order of arbitrary (not necessarily oriented) $\overrightarrow{\mathcal{O}}$-bounds.

**Lemma 7** *$K_3$ bounds $\overrightarrow{\mathcal{O}}$. $K_2$ does not bound $\overrightarrow{\mathcal{O}_g}$ for any $g$.*

**Proof:-** By the 4CT, every outerplanar graph is 3-colourable, which proves the first statement. Directed odd cycles prove the second. □

For $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \{-1, 1\}^k$ we define $L(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, N)$ to be the oriented graph comprising a directed path $0, 1, \ldots n$, and two other vertices $a$ and $b$, together with an arc from $a$ to $b$, an $\boldsymbol{\epsilon}_1$ path from $a$ to each vertex $0, 1, \ldots N$, and an $\boldsymbol{\epsilon}_2$ path from $b$ to each of these vertices. We construct $L(k, N)$ in the same way, except that we now join both $a$ and $b$ to each of the vertices $0, 1, \ldots N$ by a copy of *each* of the $2^k$ $\boldsymbol{\epsilon}$-paths of length $k$.

For each digraph $G$, we define $C(\boldsymbol{\epsilon}, G)$ to be the oriented planar graph comprising $G$, together with another vertex $a$ and an $\boldsymbol{\epsilon}$-path from $a$ to each vertex of $G$. We define $C(k, G)$ similarly by joining a vertex $a$ to every vertex of $G$ by a copy of each $\boldsymbol{\epsilon}$-path of length $k$.

**Lemma 8** *If a digraph $H$ is a minimal $\overrightarrow{\mathcal{P}_{2k+1}}$-bound, then, for each $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \{-1, 1\}^k$*

1. *For each $v \in V(H)$ the graph induced by $N^{\boldsymbol{\epsilon}}(v)$ bounds $\mathcal{O}_{2k+1}$. In particular, $|N^{\boldsymbol{\epsilon}}(v)| \geq 3$, and if $H$ is an oriented graph then $|N^{\boldsymbol{\epsilon}}(v)| \geq 7$ for $k = 1$, and $|N^{\boldsymbol{\epsilon}}(v)| \geq 5$ for $k \geq 2$.*

2. *For adjacent vertices $v, w \in V(H)$, $N^{\epsilon_1}(v) \cap N^{\epsilon_2}(w)$ induces a graph with a directed cycle (including possibly the 2-cycle, ie $K_2$). Thus this set contains at least 2 vertices and, if $H$ is an oriented graph, at least 3.*

   *If $H$ is a a minimal $\overrightarrow{\mathcal{P}_{2k}}$-bound ($k \geq 2$) then,*

3. *For each $v \in V(H)$ the graph induced by $A := \cap_{|\boldsymbol{\epsilon}|=k} N^{\boldsymbol{\epsilon}}(v)$ bounds $\mathcal{O}_{2k}$. In particular, $|A| \geq 3$, and if $H$ is an oriented graph then $|A| \geq 6$ for $k = 2$, and $|A| \geq 5$ for $k \geq 2$.*

4. *For adjacent vertices $v, w \in V(H)$, $\cap_{|\boldsymbol{\epsilon}|=k}(N^{\boldsymbol{\epsilon}}(v) \cap N^{\boldsymbol{\epsilon}}(w))$ induces a graph with a directed cycle (including possibly the 2-cycle). Thus this set contains at least 2 vertices and, if $H$ is an oriented graph, at least 3.*

**Proof:-** Let $\Gamma$ be a minimal $\overrightarrow{\mathcal{P}_{2k+1}}$-bound, $v \in V(\Gamma)$, and $O \in \overrightarrow{\mathcal{O}_{2k+1}}$. Then $C(\boldsymbol{\epsilon}, O) \in \overrightarrow{\mathcal{P}_{2k+1}}$, and so by Lemma 4, there is a homomorphism $\phi : C(\boldsymbol{\epsilon}, O) \to \Gamma$, such that $\phi(a) = v$, whence $O$ must map into $N^{\boldsymbol{\epsilon}}(v)$. Together with Theorem 6, this gives (1).

To prove (2), suppose that $v$ and $w$ are adjacent vertices in $\Gamma$. We may assume $w \in \mathrm{out}(v)$. Clearly $L(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, N) \in \overrightarrow{\mathcal{P}_{2k+1}}$, and so by Lemma 4, there is a homomorphism $\phi : L(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, N) \to \Gamma$, such that $\phi(a) = v$, $\phi(b) = w$. The vertices in $N^{\epsilon_1}(a) \cap N^{\epsilon_2}(b)$ induce a directed path of length $N$. If $N > |\Gamma|$, then $\phi$ cannot be injective on this path, whose image thus induces a digraph which contains a directed cycle in $N^{\epsilon_1}(v) \cap N^{\epsilon_2}(w)$.

The proofs of (3) and (4) proceed in exactly the same way, using the graphs $C(k, O)$ and $L(k, N)$ in place of $C(\boldsymbol{\epsilon}, O)$ and $L(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, N)$ respectively. $\square$

We now apply this lemma to begin the proof of Theorem 2. We must show that no oriented graph $H$ of order 6 or less bounds $\overrightarrow{\mathcal{P}_6}$. In fact we show that most of them cannot even bound $\overrightarrow{\mathcal{P}_8}$. We stress that this is purely a negative result; there is no claim made that any of the target graphs listed actually are bounds, only that all the other candidates are eliminated.

**Lemma 9** *1. If $H$ is an oriented $\overrightarrow{\mathcal{P}_8}$-bound of order 6 or less, then $H$ is (isomorphic to) either $T_0$ or a subgraph of $T_1$, $T_3 (= P_7^*)$, $T_4$ or $T_5$.*

*2. If $H$ is an oriented $\overrightarrow{\mathcal{P}_7}$-bound of order 6 or less, then $H$ is isomorphic to $P_7^*$*

**Proof:-** We first prove

$$\begin{array}{l}\text{No minimal oriented } \overrightarrow{\mathcal{P}_8}\text{-bound of order at most 6}\\ \text{contains a vertex with indegree or outdegree} \leq 1.\end{array} \tag{1}$$

For a contradiction we assume that $H$ is a minimal oriented $\overrightarrow{\mathcal{P}_8}$-bound of order at most 6 which does contain such a vertex $v$. Clearly $v$ cannot be a source or sink. Thus, and by symmetry, we may assume that $v$ has a single outneighbor $w$. We set $A^+ = out(w)$ and $A^- = in(w) \setminus \{v\}$.

Note that, for any vertex $x$ of $H$

$$x \notin N^{(1,1)}(\{x\}). \tag{2}$$

We have $N^{(1)}(v) = \{w\}$, whence $w \notin N^{(1,1,1)}(v)$, and so

$$v \notin N^{(1,1,1,-1)}(v). \tag{3}$$

We also have

$$N^{(1,1,1,1)}(v) = N^{(1,1)}(A^+) \tag{4}$$

$$N^{(1,1,-1,1)}(v) = N^{(-1,1)}(A^+) \tag{5}$$

$$N^{(1,-1,1,1)}(v) = A^+ \cup N^{(1,1)}\left(A^-\right) \tag{6}$$

If any of the sets on the right hand sides of (4), (5) or (6) omit any vertex other than $v$, then, by (3) and Lemma 8 (3), $H$ doesn't bound $\overrightarrow{\mathcal{P}_8}$, contrary to our assumption. Therefore we may assume that

$$N^{(1,1)}(A^+) \supseteq V(H) - \{v\} \tag{7}$$

$$N^{(-1,1)}(A^+) \supseteq V(H) - \{v\} \tag{8}$$

$$N^{(1,1)}\left(A^-\right) \supseteq A^- \cup \{w\}. \tag{9}$$

Inclusions (7) and (9), together with (2), give $|A^+| \geq 2$ and $|A^-| \geq 2$, whence $|A^+| = |A^-| = 2$. Let $A^+ = \{x, y\}$, with arc (if any) from $x$ to $y$. By (7) there is a $(1,1)$-trail which begins in $A^+$ and ends in $x$. The trail must be a path, and can only begin at $y$ and have middle vertex $m(x) \in A^-$. Similarly there is a $(1,1)$-path from $x$ to $y$ with middle vertex $m(y) \in A^-$, with $m(y) \neq m(x)$. There is also a $(1,1)$-trail which begins in $A^+$ and ends in $m(y)$. The middle vertex of this trail can only be $m(x)$ and so there must be an arc from $m(x)$ to $m(y)$. But now $m(x) \notin N^{(-1,1)}(A^+)$, which contradicts (8). This proves (1).

Since the only digraphs of order at most 6 with no vertices of indegree or outdegree at most one are $T_0$ and subgraphs of the tournaments $T_1$, $T_2$, $T_3$, $T_4$ and $T_5$, listed in Section 2, we complete the proof of the first part of the lemma, by showing that no subgraph of $T_2$ is a minimal $\overrightarrow{\mathcal{P}_8}$-bound. Let $T$ be such a subgraph of $T_2$. If 0 is a vertex of $T$, then $N^{(-1,1,1,1)}(0) \cap N^{(-1,1,1,-1)}(0) = \{1, 2, 4, 5\}$, and Lemma 8 (3) applies. Otherwise $T$ has a vertex of indegree or outdegree at most one, and we use (1)

To prove the second part of the lemma it remains to show that $T_0$, the subgraphs of $T_1$, $T_4$ and $T_5$ and the proper subgraphs of $T_3 = P_7^*$ do not bound $\overrightarrow{\mathcal{P}_7}$. For $T_0$, $N^{(1,1,1)}(0) = \{0, 1, 3, 4\}$, so Lemma 8(1) shows it is not a minimal-$\overrightarrow{\mathcal{P}_7}$-bound.

We claim that a digraph $T$ of order 6 does not bound $\overrightarrow{\mathcal{P}_7}$, if it contains an arc $(u, v)$ with either $d^+(u) \leq 2$ or $d^-(v) \leq 2$, and, for some $\pm 1$ sequences $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ of length 3, $N^{\boldsymbol{\epsilon}_1}(u) \cap N^{\boldsymbol{\epsilon}_2}(v)$ induces a graph with no cycle. It suffices to show that no subgraph $T'$ of $T$ is a *minimal* $\overrightarrow{\mathcal{P}_7}$-bound. Either $|T'| \leq 5$, $T'$ has a vertex of indegree or outdegree at most one or $u$ and $v$ are adjacent in $T'$. In the first two cases we have already shown that $T'$ does not bound $\overrightarrow{\mathcal{P}_7}$, and in the last, Lemma 8 (2) gives the same conclusion. This proves the claim.

For $T = T_1$ we let $(u, v) = (5, 0)$ and compute $N^{(1,-1,-1)}(5) \cap N^{(-1,1,-1)}(0) = \{2, 3, 4, 5\}$; for $T = T_4$, we let $(u, v) = (1, 2)$ and compute $N^{(1,1,1)}(1) \cap N^{(-1,1,-1)}(2) = \{1, 3, 4, 5\}$. Since in each case the vertex set induces a transitive $K_4$, neither of these tournaments bound $\overrightarrow{\mathcal{P}}_7$, by the claim above. Since $T_5$ is anti-isomorphic to $T_4$, Lemma 2 now shows that $T_5$ does not bound $\overrightarrow{\mathcal{P}}_7$ either.

By Lemma 5 every proper subgraph of $T_3 = P_7^*$ either has a vertex of indegree or outdegree $\leq 1$ or is isomorphic a subgraph of $T_4$ or $T_5$, so that either way we are already done. $\square$

In the remaining case, $T = T_3$, Lemma 8 is of no value, since every pair of distinct vertices is joined by all 8 kinds of trail of length 3. We need another approach. The following lemma follows directly from the definitions.

**Lemma 10** *If there is a homomorphism from $G$ to $H$, then there is a homomorphism from $G^k$ to $H^k$.*

Indeed if $\varphi : G \to H$ is a homomorphism, then the very same map (regarded as a function of vertices) is also a homomorphism from $G^k$ to $H^k$. This lemma is most useful when $k = 2$, since the square of an oriented graph has no loops. We use

**Corollary 11** *If $H$ bounds $\overrightarrow{\mathcal{P}_{2k}}$, then $H^2$ bounds $\overrightarrow{\mathcal{P}}_k$.*

**Proof:-** Suppose that $H$ bounds $\mathcal{P}_{2k}$, and let $G \in \mathcal{P}_k$. Construct $\tilde{G}$ from $G$ by replacing each arc of $G$ by a directed path of length 2. Thus $\tilde{G} \in \mathcal{P}_{2k}$, whence there is a homomorphism from $\tilde{G}$ to $H$, and so from $\tilde{G}^2$ to $H^2$. But one of the components of $\tilde{G}^2$ is $G$. Thus $G$ has a homomorphism to $H^2$, so that $H^2$ bounds $\mathcal{P}_k$. $\square$

Through this result we move naturally from considering oriented target graphs to target graphs with opposite arcs. It is easy to check that $(P_7^*)^2$ has an arc from $n$ to $m$, except in the case where $m$ is a quadratic residue and $m = -n$. That is $U := (P_7^*)^2$ is the complement of a 6-vertex digraph with 3 mutually non-incident arcs. We complete the proof of Theorem 2 by showing

**Lemma 12** *$U$ does not bound $\overrightarrow{\mathcal{P}}$*

**Proof:-** We show that if $U'$ is a subgraph of $U$ then $U$ is not a minimal $\overrightarrow{\mathcal{P}}$-bound. If $|U'| \leq 3$, this follows from Lemma 8 (1). Otherwise $U'$ contains a quadratic residue $u$ and a non-quadratic residue $v \neq -u$, and $in(u) \cap out(v) = \{1,2,3,4,5,6\} \setminus \{u,-u,v,-v\}$, which induces an oriented $K_2$. Now Lemma 8(2) shows that $U'$ is not a minimal $\overrightarrow{\mathcal{P}}$-bound. $\Box$

Note that $U$ contains no (unoriented) $K_4$. If it did so then it would have to bound $\mathcal{P}$ by the 4CT. We now turn to the problem of finding small $K_4$-free $\mathcal{P}$-bounds. We first introduce some notation.

**Definition:-** $T(t,i)$ is the complement of the disjoint union of $t$ directed triangles and $i$ isolated vertices. In other words $T(t,i)$ is obtained by removing $t$ vertex disjoint directed triangles from $K_{3t+i}$. We will let $C_1, C_2, \ldots C_t$ denote the vertex sets of these triangles, and $s_1, s_2, \ldots s_i$ the remaining vertices.

**Lemma 13** 1. *$T(t,i)$ is $K_4$-free if and only if $t+i<4$.*

2. *Every oriented $2t+i-1$-degenerate graph $\Gamma$ is $T(t,i)$-colourable.*

**Proof:-** The first part is trivial, and the second is proved by induction on $|\Gamma|$. For the induction step, let $G$ be a $2t+i-1$-degenerate graph, $v$ a vertex of $G$ of order at most $2t+i-1$ and $\phi$ a $T(t,i)$-colouring of $G-\{v\}$. Let $N$ be the set of colours taken by the neighbours of $v$. Either some $s_k \notin N$, in which case we use this vertex to colour $v$, or else some $C_k$ contains at most one colour in $N$, and this is used to colour only one neighbour of $v$. In this case we may colour $v$ by one of the other vertices in $C_k$. $\Box$

An immediate corollary of this is that $T(3,0)$, of order 9, bounds the class of oriented 5-degenerate–and hence planar–graphs. This seems to be the smallest bound that is really easy to prove, but we can do better.

**Theorem 14** *1. $T(2,1)$ bounds $\overrightarrow{\mathcal{P}}$*

*2. $T(1,1)$ bounds $\overrightarrow{\mathcal{P}_5}$*

We prove this in the next section.

Theorem 14 naturally raises the following problems.

**Problem 1** *Does $T(1,2)$ bound $\overrightarrow{\mathcal{P}}$? Does $T(1,1)$ bound $\overrightarrow{\mathcal{P}_4}$?*

We note that $T(1,2)$ is the only digraph $H$ of order $\leq 5$ that neither contains a $K_4$ nor is excluded by Lemma 8 from being a $\overrightarrow{\mathcal{P}}$-bound. We outline the proof; the reader may fill in the details. If $|H| \leq 4$, then either $H$ is isomorphic to $K_4$ or Lemma 8 shows that $H$ doesn't bound $\mathcal{P}$, so we suppose that $|H| = 5$. If $\overline{H}$ has two arcs without a common endpoint or three arcs in a transitive triangle, then Lemma 8 (2) shows that $H$ doesn't bound $\mathcal{P}$; if the arcs in $\overline{H}$ have a common endpoint, then $H$ contains $K_4$. The only remaining possibility is that the arcs of $\overline{H}$ form a directed triangle, *ie* $H = T(1,2)$.

A similar argument shows that $T(1,1)$ is the only digraph $H$ of order $\leq 4$ that neither contains a $K_3$, nor is excluded by Lemma 8 from being a $\overrightarrow{\mathcal{P}_5}$-bound.

# 4 List homomorphisms

We now modify two of Thomassen's list coloring theorems [17, 20], in order to prove Theorem 14. First we generalize his proof of the 5-choosability of planar graphs [17].

**Theorem 15** *Let $H$ be a digraph, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of vertex sets such that For every $B \in \mathcal{B}$, $u \in V(H)$ and $i \in \{-1,1\}$, $B \cap N^{(i)}(u)$ contains a subset $B'$ for which*

1. *For all $v \in V(H)$ and $j \in \{-1,1\}$, $B' \cap N^{(j)}(v) \neq \emptyset$*
2. *For all $A \in \mathcal{A}$ and $j \in \{-1,1\}$, $A \cap_{w \in B'} N^{(j)}(u) \in \mathcal{B}$*

*Let $G$ be an oriented plane graph, $(v_1, v_2)$ be an arc on the outer face boundary and $c$ be a homomorphism of $(v_1, v_2)$ to $H$. For each vertex $v$ in $G$, let $L(v) \subseteq V(H)$. If $v = v_1, v_2$ then $L(v)$ consists of $c(v)$, $L(v) \in \mathcal{B}$ if $v \neq v_1, v_2$ is on the outer face boundary and $L(v) \in \mathcal{A}$ in all other cases. Then $c$ can be extended to a homomorphism $\tilde{c} : G \to H$, such that $\tilde{c}(v) \in L(v)$.*

We get the statement of Theorem 15 from Thomassen's original theorem by substituting sets in $\mathcal{A}$ and in $\mathcal{B}$ for 5-sets and 3-sets respectively. His proof then also applies, with the same modifications. We may recover the original result from Theorem 15 by letting $H$ be the complete graph on the union of all lists, and $\mathcal{A}$ and $\mathcal{B}$ comprise the 5-sets and 3-sets respectively.

We can now prove Theorem 14 (1), by applying Theorem 15 with $H = T(2, 1)$, $\mathcal{A} = \{V(H)\}$ and $\mathcal{B}$ is the collections of vertex sets which contain either $s_1$ and one point from each of $C_1$ and $C_2$, or one of the sets $C_1$, $C_2$ and at least one other point. We can then always choose $B'$ above to be either $C_1$, $C_2$ or a 2-set contained in neither of these triangles.

Let $H$ be a digraph. Call a subset $S$ of $V(H)$ *good* if, for any two (not necessarily distinct) vertices $u$ and $v$ of $H$ and any $i, j \in \{-1, 1\}$, $S \cap N^{(i)}(u) \cap N^{(j)}(v) \neq \emptyset$.

**Theorem 16** *Let $H$ be a digraph, and $G \in \overrightarrow{\mathcal{P}_5}$. Let $c$ be a homomorphism of a path or cycle $P : v_1 v_2 \ldots v_q$, $1 \leq q \leq 6$, such that all the vertices of $P$ are on the outer face boundary. For each vertex $v$ in $G$, let $L(v) \subseteq V(H)$. If $v \in P$ then $L(v)$ consists of $c(v)$. Otherwise $L(v)$ contains a set of the form $S \cap N^{(i)}(u)$, where $S$ is good. If $v$ is not on the outer face boundary, then $L(v)$ is good. Assume furthermore that if an edge, not in $P$, joins vertices $u$ and $v$, then either $L(u)$ and $L(v)$ are both good. Then $c$ can be extended to a homomorphism $\tilde{c} : G \to H$, such that $\tilde{c}(v) \in L(v)$.*

This is an almost verbatim restatement of Thomassen's list version of Grötzsch's theorem [20] (Theorem 2.1). Where in the original theorem we have 3-sets and 2-sets, we now have respectively good sets and sets of the form $S \cap N^{(i)}(u)$, where $S$ is good. The proof in [20] then applies with the same modifications. (Again we can recover the original list coloring theorem in [20] by setting $H$ to be the complete graph on the union of all the lists, and taking 3-sets as the good sets).

In the next result, the hypothesis is that $V(H)$ itself is good.

**Corollary 17** *Let $H$ be a digraph with the property that, for any two (not necessarily distinct) vertices $u$ and $v$ of $H$ and any $i, j \in \{-1, 1\}$, $N^{(i)}(u) \cap N^{(j)}(v) \neq \emptyset$, then $H$ bounds $\overrightarrow{\mathcal{P}_5}$.*

Setting $H = T(1,1)$ above proves Theorem 14 (2). Here is another application. Let $W := T_0^2 = C(5;1,2)^2$. It is easy to check that $W$ is the complement of a directed 5-cycle. Corollary 17 thus gives

**Theorem 18** *$W$ bounds $\overrightarrow{\mathcal{P}_5}$.*

We conjecture two stronger results.

**Conjecture 1** *$W$ bounds all oriented graphs with maximum average degree $< 10/3$.*

**Conjecture 2** *$C(5;1,2)$ bounds $\overrightarrow{\mathcal{P}_{10}}$.*

To see that these are indeed generalizations of Theorem 18, we use respectively the well known fact that a graph in $\mathcal{P}_g$ has maximum average degree $< 2g/(g-2)$ (see eg [10]), and Corollary 11.

Here is yet another possible strengthening of Theorem 18.

**Problem 2** *Does $W$ bound $\overrightarrow{\mathcal{P}}_4$?*

Lemma 8 (2) shows that $W$ does not bound $\overrightarrow{\mathcal{P}}$. For a more direct approach, let $G$ be the square pyramid with directed base with an extra arc joining two diagonally opposite vertices. If the arcs are directed from the apex to each vertex of the base, then $G$ admits no homomorphism to $W$. We leave it as an exercise to show this.

## 5 Another proof of Theorem 14

We can also prove Theorem 14 using the following theorem. The first two parts were conjectured by Borodin [1], and proved by Thomassen in [18] and [19], respectively; the third is due to Borodin and Glebov [2].

**Theorem 19 ([2, 18, 19])** *1. The vertices of a planar graph can be partitioned into two sets $V_1$ and $V_2$ which induce respectively a 1-degenerate graph (ie a forest) and a 2-degenerate graph.*

*2. The vertices of a planar graph can be partitioned into two sets $V_1$ and $V_2$ which induce respectively a 0-degenerate graph (ie an independent set) and a 3-degenerate graph.*

*3. The vertices of a planar graph of girth at least 5 can be partitioned into two sets $V_1$ and $V_2$ which induce respectively a 0-degenerate and a 1-degenerate graph.*

**Proof of Theorem 14:-** (1) The vertices of $T(2,1)$ can be partitioned into sets $A$ and $B$, where $A$ and $B$ spans a $K_0$ and a $T(2,0)$ respectively. Moreover the single vertex in $A$ joins each vertex in $B$ by arcs in both directions.

Let $G$ be an oriented planar graph, and let $V_1$, $V_2$ be a vertex partition as in (2) of the above theorem. By Lemma 13 we can colour the vertices of the graphs induced by $V_1$ and $V_2$ from $A$ and $B$, respectively. This gives a $T(2,1)$-colouring of $G$. (We could equally well partition $T(2,1)$ into a $T(1,1)$ and a directed triangle, and use (1) of Theorem 19.) (2) Similar argument. Details omitted. $\square$

## References

[1] O.V. Borodin, On decomposition of graphs into degenerate subgraphs, *Discret Analiz.* **28** (1976) 3-11.

[2] O.V. Borodin, A.N. Glebov. On the partition of a planar graph of girth 5 into an empty graph and an acyclic subgraph, *Diskretn. Anal. Issled. Oper. Ser. 1* **8** (2001) 34-53 (in Russian).

[3] Borodin and A. O. Ivanova. An oriented 7-colouring of planar graphs with girth at least 7. *Sib. Electron. Math. Reports*, **2** 222-229, 2005.

[4] O. V. Borodin, A. O. Ivanova, and A. V. Kostochka. Oriented 5-coloring of sparse plane graphs. *J. Applied and Industrial Mathematics*, 1(1):9-17, 2007.

[5] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena On the maximum average degree and the oriented chromatic number of a graph, *Discrete Math.* **206** (1999) 77-90.

[6] P. Hell and J. Nešetřil, Graphs and Homomorphisms, Oxford Lecture Series in Mathematics and its Applications, 28, Oxford University Press (2004)

[7] T. H. Marshall Homomorphism bounds for oriented planar graphs, *J. Graph Theory* **55** (2007) 175-190.

[8] T. H. Marshall On oriented graphs with certain extension properties, To appear in *Ars Combinatoria*

[9] T. H. Marshall An oriented 6-coloring of planar graphs with girth at least 9, To be submitted.

[10] J. Nešetřil, A. Raspaud and E. Sopena, Colorings and girth of oriented planar graphs, *Discrete Math.* **165-166** (1997) 519-530.

[11] P. Ochem. Oriented colorings of triangle-free planar graphs. *Inform. Process. Lett.* **92** (2004) 71-76.

[12] P. Ochem and A. Pinlou, Oriented coloring of triangle-free planar graphs and 2-outerplanar graphs *Elect. notes in Discrete Math.* **37** (2011) 123-128

[13] A. Pinlou, An oriented coloring of planar graphs with girth at least five, *Discrete Math.*, **309** (2009) 2108-2118

[14] A. Pinlou and E. Sopena, Oriented vertex and arc colorings of outerplanar graphs *Inform. Proc. Letters* **100** (2006) 97-104

[15] A. Raspaud and E. Sopena, Good and semi-strong colorings of oriented planar graphs, *Inform. Proc. Letters* **51** (1994) 171-174.

[16] E. Sopena, There exist oriented planar graphs with oriented chromatic number at least sixteen, *Inform. Proc. Letters* **81** (2002) 309-312.

[17] C. Thomassen, Every planar graph is 5-choosable, *J. Combin. Theory Ser. B* **62** (1994) 180-181.

[18] C. Thomassen, Decomposing a planar graph into degenerate graphs, *J. Combin. Theory Ser. B* **65** (1995) 305-314.

[19] C. Thomassen, Decomposing a planar graph into an independent set and a 3-degenerate graph, *J. Combin. Theory Ser. B* **83** (2001) 262-271.

[20] C. Thomassen, A short list color proof of Grötzsch's theorem, *J. Combin. Theory Ser. B* **88** n. 1 (2003) 189-192.