A new lower bound based on Gromov's method of selecting heavily covered points*

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Abstract

Boros and Füredi (for $d = 2$) and Bárány (for arbitrary d) proved that there exists a positive real number c_d such that for every set P of n points in $\mathbf{R}^d$ in general position, there exists a point of $\mathbf{R}^d$ contained in at least $c_d\binom{n}{d+1}$ $(d+1)$-simplices with vertices at the points of P. Gromov improved the known lower bound on c_d by topological means. Using methods from extremal combinatorics, we improve one of the quantities appearing in Gromov's approach and thereby provide a new stronger lower bound on c_d for arbitrary d. In

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particular, we improve the lower bound on c_3 from 0.06332 to more than 0.07509; the best upper bound known on c_3 being 0.09375.

1 Introduction

We study an extremal graph theory problem linked to a classical geometric problem through a recent work of Gromov [8]. The geometric result that initiated this work is a theorem of Bárány [2], which extends an earlier generalization of Carathéodory's theorem due to Boros and Füredi [4].

Theorem 1 (Bárány [2]). *Let d be a positive integer. There exists a positive real number c such that for every set P of points in $\mathbf{R}^d$ that are in general position, there is a point of $\mathbf{R}^d$ that is contained in at least*

$$c \cdot \binom{|P|}{d+1} - O\Big(|P|^d\Big) \tag{1}$$

d-dimensional simplices spanned by the points in P.

Define c_d to be the supremum of all the real numbers that satisfy (1) in Theorem 1 for the dimension d.

Bukh, Matoušek and Nivasch [6] established that

$$c_d \leqslant \frac{(d+1)!}{(d+1)^{d+1}}$$

by constructing suitable configurations of n points in $\mathbf{R}^d$. On the lower bound side, Boros and Füredi [4] proved that $c_2 \geqslant 2/9$ which matches the upper bound; so $c_2 = 2/9$ (another proof was given by Bukh [5]). Bárány's proof [2] yields that $c_d \geqslant (d+1)^{-d}$. Wagner [18] improved this lower bound to

$$c_d \geqslant \frac{d^2+1}{(d+1)^{d+1}}.$$

Further improvements of the lower bound for c_3 were established by Basit et al. [3] and by Matoušek and Wagner [15].

Gromov [8] developed a topological method for establishing lower bounds on c_d (Matoušek and Wagner [15] provided an exposition of the combinatorial components of his method, while Karasez [13] managed to simplify Gromov's approach). His method yields a bound that matches the optimal bound for $d = 2$ and is better than that of Basit et al. [3] for $d = 3$. We need several definitions to state Gromov's lower bound. Fix a positive integer d and a finite set V. A *d-system E on V* is a family of d-element subsets of V. The *density* of the system E is $\|E\| := |E| / \binom{|V|}{d}$. The *coboundary* δE of a d-system E on V is the $(d+1)$-system composed of those $(d+1)$-element subsets of V that contain an odd number of sets of E. The coboundary operator δ commutes with the symmetric difference, i.e., $\delta(A \triangle B) = (\delta A) \triangle (\delta B)$. It is not hard to show that $\delta\delta E = \mathbf{0}$ for any d-system E where $\mathbf{0}$ is the empty $(d+2)$-system. In fact, the converse also holds: a d-system E is a coboundary of a $(d-1)$-system if and only if $\delta E = \mathbf{0}$.

A d-system E on V is *minimal* if $\|E\| \leqslant \|E'\|$ for any d-system E' on V with $\delta E = \delta E'$. This is equivalent to saying that $\|E\| \leqslant \|E \triangle \delta D\|$ for every $(d-1)$-system D on V. Let $\mathcal{M}_d(V)$ be the set of all minimal d-systems on V and define the following functions:

$$\varphi_d(\alpha) := \liminf_{|V| \to \infty} \min \{ \|\delta E\| \mid E \in \mathcal{M}_d(V) \text{ and } \|E\| \geqslant \alpha \} .$$

It is easy to observe that the functions φ_d are defined for $\alpha \in [0, 1/2]$ and $\varphi_1(\alpha) = 2\alpha(1-\alpha)$. It can also be shown that $\varphi_d(\alpha) \geqslant \alpha$.

Gromov's lower bound on the quantity c_d is given in the next theorem.

Theorem 2 (Gromov [8]). *For every positive integer d, it holds that*

$$c_d \geqslant \varphi_d\left(\frac{1}{2}\varphi_{d-1}\left(\frac{1}{3}\varphi_{d-2}\left(\cdots\frac{1}{d}\varphi_1\left(\frac{1}{d+1}\right)\cdots\right)\right)\right) . \qquad (2)$$

Plugging the bound $\varphi_d(\alpha) \geqslant \alpha$ in (2), we obtain

$$c_d \geqslant \frac{2d}{(d+1)!(d+1)}. \tag{3}$$

Improvements of the bound in (3) can be obtained by proving stronger lower bounds on the functions φ_d. The first step in this direction has been done by Matoušek and Wagner.

Theorem 3 (Matoušek and Wagner [15])**.**

- *For all* $\alpha \in [0\,, 1/4]$*, it holds that*

$$\varphi_2(\alpha) \geqslant \frac{3}{4}\left(1 - \sqrt{1-4\alpha}\right)(1-4\alpha).$$

- *For all sufficiently small* $\alpha > 0$*, it holds that*

$$\varphi_3(\alpha) \geqslant \frac{4}{3}\alpha - O(\alpha^2).$$

Our main result asserts a stronger lower bound on $\varphi_2(\alpha)$ for $\alpha \in [0\,, 2/9]$ which are the values appearing in Theorem 2.

Theorem 4. *For all* $\alpha \in [0\,, 2/9]$*, it holds that*

$$\varphi_2(\alpha) \geqslant \frac{3}{4}\alpha(3 - \sqrt{8\alpha+1}).$$

When plugged into Theorem 2, our bound yields that $c_3 > 0.07433$. For comparison, the earlier bounds of Wagner [18], Basit et al. [3], Gromov [8] and Matoušek and Wagner [15] are $c_3 \geqslant 0.03906$, $c_3 \geqslant 0.05448$, $c_3 \geqslant 0.0625$ and $c_3 \geqslant 0.06332$, respectively. The proof of Matoušek and Wagner [15] is based on analysing combinatorial objects called *pagodas*. This analysis involves the functions φ_1, φ_2 and φ_3 (see [15, Proposition 15]). So, we can refine the analysis by using the new lower bound on φ_2 stated in Theorem 4, thereby improving the value of ϵ_0 in the statement of Proposition 15 in [15] from 0.00082 to 0.012589. This yields a yet better bound on c_3, namely $c_3 > 0.07509$.

The definition of the function φ_2 can naturally be cast in the language of graphs. A *cut* of a graph G is a partition of vertices of G into two (disjoint) parts; a (non-)edge that cross the partition is said to be *contained* in the cut. A graph is *Seidel-minimal* if no cut contains more edges than non-edges. It is straightforward to see that a graph G with vertex set V is Seidel-minimal if and only if its edge-set viewed as a 2-system is minimal. Let $\mathcal{S}_n(\alpha)$ be the set of all Seidel-minimal graphs on n vertices with density at least α, i.e., with at least $\alpha\binom{n}{2}$ edges. Further, let $\mathcal{S}(\alpha)$ be the union of all $\mathcal{S}_n(\alpha)$.

A triple T of vertices of a graph G is *odd* if the subgraph of G induced by T contains precisely either one or three edges. Finally, let $\varphi_g(G)$ for a graph G be the density of odd triples in G, i.e.,

$$\varphi_g(G) = \frac{\left|\left\{T \in \binom{V(G)}{3} \,\middle|\, T \text{ is odd}\right\}\right|}{\binom{|V(G)|}{3}}.$$

It is not hard to show that for every $\alpha \in [0\,, 1/2]$,

$$\varphi_2(\alpha) = \liminf_{n\to\infty} \min\left\{\varphi_g(G) \mid G \in \mathcal{S}_n(\alpha)\right\}.$$

Using this reformulation to the language of graph theory, we show that $\varphi_2(\alpha) \geqslant \frac{3}{4}\alpha(3 - \sqrt{8\alpha+1})$ for $\alpha \in [0\,, 2/9]$. Our proof is based on the notion of flag algebras developed by Razborov [16], which builds on the work of Lovász and Szegedy [14] on graph limits and of Freedman et al. [7]. The notion was further applied, e.g., in [1, 9–12, 17]. We do not use the full strength of this notion here and we survey the relevant parts in Section 2 to make the paper as much self-contained as possible. In Section 3, we provide a bound $\varphi_2(\alpha) \geqslant \frac{9}{7}\alpha(1-\alpha)$ using just some of the methods presented in Section 2. The purpose of this section is to get the reader acquainted with the notation. Our main result is proved in Section 4.

2 Flag algebras

In this section, we review some of the theory related to flag algebras which were introduced by Razborov [16]. We focus on the concepts that are relevant to our proof. The reader is referred to the seminal paper of Razborov [16] for a complete and detailed exposition of the topic.

Fix $\alpha > 0$ and consider a sequence of graphs $(G_i)_{i\in\mathbf{N}}$ from $\mathcal{S}(\alpha)$ such that

$$\lim_{i\to\infty} |V(G_i)| = \infty \quad \text{and} \quad \lim_{i\to\infty} \varphi_g(G_i) = \varphi_2(\alpha).$$

Let $p(H, H_0)$ be the probability that a randomly chosen subgraph of H_0 with $|V(H)|$ vertices is isomorphic to H. The sequence G_i must contain a subsequence $(G_{i_j})_{j\in\mathbf{N}}$ such that $\lim_{j\to\infty} p(H, G_{i_j})$ exists for every graph H. Define $q_\alpha(H) \coloneqq \lim_{j\to\infty} p(H, G_{i_j})$. Observe that the definition of q_α implies that $q_\alpha(K_2) \geqslant \alpha$ and $q_\alpha\left(\overline{P_3}\right) + q_\alpha(K_3) = \varphi_2(\alpha)$ where $\overline{P_3}$ is the complement of the 3-vertex path.

The values of $q_\alpha(H)$ for various graphs H are highly correlated. Let $\mathcal{F}$ be the set of all graphs and $\mathcal{F}_\ell$ the set of graphs with ℓ vertices. Extend the mapping $q_\alpha(H)$ from $\mathcal{F}$ to $\mathbf{R}\mathcal{F}$ by linearity, where $\mathbf{R}\mathcal{F}$ is the linear space of formal linear combinations of the elements of $\mathcal{F}$. Next, let $\mathcal{K}$ be the subspace of $\mathbf{R}\mathcal{F}$ generated by the elements of the form

$$H_0 - \sum_{H\in\mathcal{F}_\ell} p(H_0, H)H$$

for all graphs H_0 and all $\ell > |V(H_0)|$. It can be shown (and it is not hard) that the mapping $q_\alpha : \mathbf{R}\mathcal{F} \to \mathbf{R}$ is consistent with $\mathcal{K}$, that is, $q_\alpha(F) = q_\alpha(F + F')$ for every $F \in \mathbf{R}\mathcal{F}$ and $F' \in \mathcal{K}$.

Let $p(H_1, H_2; H_0)$ be the probability that two randomly chosen disjoint subsets V_1 and V_2 with cardinalities $|V(H_1)|$ and $|V(H_2)|$ induce in H_0 subgraphs isomorphic to H_1 and H_2, respectively. For two graphs H_1 and H_2, define their product to be

$$H_1 \times H_2 \coloneqq \sum_{H_0\in\mathcal{F}_\ell} p(H_1, H_2; H_0)H_0$$

where $\ell = |V(H_1)|+|V(H_2)|$. The product operator can be extended to $\mathbf{R}\mathcal{F} \times \mathbf{R}\mathcal{F}$ by linearity. Since the product operator defined in this way is consistent with the equivalence relation on the elements of $\mathbf{R}\mathcal{F}$ induced by $\mathcal{K}$, we can consider the quotient $\mathcal{A} := \mathbf{R}\mathcal{F}/\mathcal{K}$ as an algebra with addition and multiplication. Since q_α is consistent with $\mathcal{K}$, the function q_α naturally gives rise to a mapping from $\mathcal{A}$ to $\mathbf{R}$, which is in fact a homomorphism from $\mathcal{A}$ to $\mathbf{R}$. In what follows, we use q_α for this homomorphism exclusively. To simplify our notation, we will use $q_\alpha(F)$ for $F \in \mathbf{R}\mathcal{F}$ but we also keep in mind that F stands for a representative of the equivalence class of $\mathbf{R}\mathcal{F}/\mathcal{K}$.

A homomorphism $q : \mathcal{A} \to \mathbf{R}$ is *positive* if $q(F) \geqslant 0$ for every $F \in \mathcal{F}$. Positive homomorphisms are precisely those corresponding to the limits of convergent graph sequences. We write $F \geqslant 0$ for $F \in \mathcal{A}$ if $q(F) \geqslant 0$ for any positive homomorphism q. Such $F \in \mathcal{A}$ form the semantic cone $\mathcal{C}_{\text{sem}}(\mathcal{A})$. Razborov [16] developed various general and deep methods for proving that $F \geqslant 0$ for $F \in \mathcal{A}$. Here, we will use only one of them, which we now present.

Consider a graph σ and let $\mathcal{F}^\sigma$ be the set of graphs G equipped with a mapping $\nu : \sigma \to V(G)$ such that ν is an embedding of σ in G, i.e., the subgraph induced by the image of ν is isomorphic to σ. We can extend the definitions of the quantities $p(H, H_0)$ and $p(H_1, H_2; H_0)$ to this "labelled" case by requiring that the randomly chosen sets always include the image of ν and preserve the mapping ν. Similarly as before, one can define $\mathcal{K}^\sigma$, $\mathcal{A}^\sigma = \mathbf{R}\mathcal{F}^\sigma/\mathcal{K}^\sigma$, positive homomorphisms, etc.

The intuitive interpretation of homomorphisms from $\mathcal{A}^\sigma$ to $\mathbf{R}$ is as follows: for a fixed embedding ν of σ, the value $q_\nu(F)$ for $F \in \mathcal{F}^\sigma$ is the probability that a randomly chosen superset of the image of ν is isomorphic to F. It can be shown that a positive homomorphism q from $\mathcal{A}$ to $\mathbf{R}$ gives rise to a unique probability distribution on positive homomorphisms q^σ from $\mathcal{A}^\sigma$ to $\mathbf{R}$ such that this probability distribution is the limit of the probability distributions of q_ν given by random choices of ν in the graphs in any convergent sequence corresponding to q.

Define $[\![H]\!]_\sigma$ for $H \in \mathcal{F}^\sigma$ to be $p \cdot H'$ where H' is the unlabeled

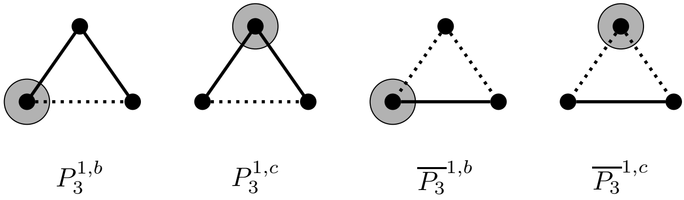


Figure 1: Four elements of $\mathcal{A}^1$.

version of H and p is the probability that a randomly chosen mapping ν from σ to the graph H is an embedding of σ that yields H as the element of $\mathcal{F}^\sigma$. The operator $[\![\cdot]\!]_\sigma$ is extended from $\mathcal{F}^\sigma$ to $\mathcal{A}^\sigma$ by linearity. For a positive homomorphism q from $\mathcal{A}$ to $\mathbf{R}$, one can think of the value of $q([\![F]\!]_\sigma)$ for $F \in \mathcal{A}^\sigma$ as the expected value of $q^\sigma(F)$ with respect to the probability distribution on q^σ corresponding to q. Hence, if $q^\sigma(F) \geqslant 0$ with probability one, then $q([\![F]\!]_\sigma) \geqslant 0$.

2.1 Example

As an example of the introduced formalism, we prove that $\varphi_2(\alpha) \geqslant \alpha$. The following notation is used: K_n is the complete graph with n vertices, P_n is the n-vertex path and $\overline{K}_n$ and $\overline{P}_n$ are their complements, respectively. We also use 1 for K_1 to simplify the notation. The following elements of $\mathcal{A}^1$ will be of particular interest to us: $P_3^{1,b}$ is P_3 with 1 embedded to the end vertex of the path and $P_3^{1,c}$ is P_3 with 1 embedded to the central vertex; $\overline{P_3}^{1,b}$ and $\overline{P_3}^{1,c}$ are their complements, respectively. See Figure 1 for an illustration of this notation.

Consider the homomorphism q_α from $\mathcal{A}$ to $\mathbf{R}$. Recall that $\alpha \leqslant q_\alpha(K_2)$. Since it holds that

$$K_2 - \frac{1}{3}\overline{P_3} - \frac{2}{3}P_3 - K_3 \in \mathcal{K},$$

we obtain

$$\alpha \leqslant q_\alpha \left(\frac{1}{3}\overline{P_3} + \frac{2}{3}P_3 + K_3 \right). \tag{4}$$

We now use that the graphs in the sequence defining q_α are Seidel-minimal. Let G_i be a graph in this sequence, n the number of its vertices and v an arbitrary vertex of G_i. Let A be the neighbors of v and B its non-neighbors. Since G is Seidel-minimal, the number of edges between A and B does not exceed the number of non-edges between A and B (increased by $O(n)$ for the inclusion of v in one or the other side of the cut; however, this term will vanish in the limit). Therefore,

$$0 \leqslant q_\alpha \left(\left[\!\left[\overline{P_3}^{1,b} - P_3^{1,b} \right]\!\right]_1 \right). \tag{5}$$

Applying the operator $[\![\cdot]\!]_1$ in (5) yields that

$$0 \leqslant q_\alpha \left(\frac{2}{3}\overline{P_3} - \frac{2}{3}P_3 \right). \tag{6}$$

Summing (4) and (6) (recall that q_α is a homomorphism from $\mathcal{A}$ to $\mathbf{R}$), we obtain

$$\alpha \leqslant q_\alpha \left(\overline{P_3} + K_3 \right) = \varphi_2(\alpha).$$

This completes the proof.

A similar argument applied to the algebra based on d-uniform hypergraphs yields that $\varphi_d(\alpha) \geqslant \alpha$. However, since we do not want to introduce additional notation not necessary for the exposition in the rest of the paper, we omit further details.

3 First bound

To become more acquainted with the method, we now present a bound that is both weaker and simpler than our main result. Fix the enumeration of 4-vertex graphs as in Figure 2. To simplify our formulas, $q_\alpha \left(\sum_{i=1}^{11} \xi_i F_i \right)$ shall simply be written $q_\alpha\left(\xi_1, \cdots, \xi_{11}\right)$.

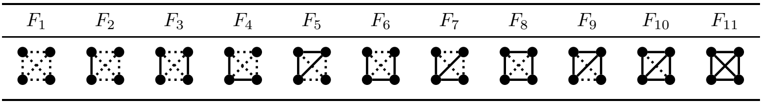


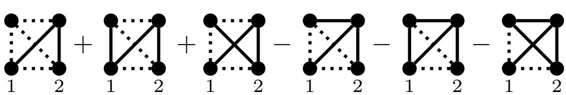
Figure 2: The eleven non-isomorphic graphs with 4 vertices.

Theorem 5. *For every $\alpha \in [0\,,2/9]$, it holds that $q_\alpha\left(\overline{P_3} + K_3\right) \geqslant \frac{9}{7}\alpha(1-\alpha)$.*

Proof. We first establish three inequalities on the values taken by q_α for various elements of $\mathcal{A}$. The choice of the graphs in the sequence defining q_α implies that $\alpha \leqslant q_\alpha\left(K_2\right)$. As $q_\alpha\left(\overline{K}_2\right) = 1 - q_\alpha\left(K_2\right)$ and $q_\alpha(K_2) \in [0\,,1/2]$, we infer that

$$\alpha(1-\alpha) \leqslant q_\alpha\left(K_2\right) q_\alpha\left(\overline{K}_2\right) = q_\alpha\left(K_2 \times \overline{K}_2\right) = q_\alpha\left(0, \frac{1}{6}, 0, \frac{1}{3}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2}, 0, \frac{1}{3}, \frac{1}{6}, 0\right). \tag{7}$$

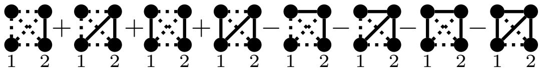
The other two inequalities follow from the Seidel minimality of graphs in the sequence defining q_α. Consider a graph G_i and two non-adjacent vertices v_1 and v_2. Let A be the set of their common neighbors and B the set of the remaining vertices. Applying the Seidel minimality to the cut given by A and B, we obtain the following inequality.

$$0 \leqslant q_\alpha\Bigg(\Bigg[\!\!\Bigg[$$

$$\Bigg]\!\!\Bigg]_{\overline{K}_2}\Bigg).$$

Evaluating the operator $[\![\cdot]\!]_{\overline{K}_2}$ yields that

$$0 \leqslant q_\alpha\left(0, 0, 0, \frac{1}{6}, 0, \frac{1}{3}, -\frac{1}{2}, 0, -\frac{1}{3}, 0, 0\right). \tag{8}$$

Now, let A' be the neighbors of v_2 and B' its non-neighbors. The Seidel minimality of cuts of this type yields that

$0 \leqslant q_\alpha \Big([\![$  $]\!]_{\overline{K_2}} \Big)$,

which subsequently implies that

$$0 \leqslant q_\alpha \left(0, \frac{1}{3}, \frac{2}{3}, 0, 0, 0, -\frac{1}{2}, 0, -\frac{1}{6}, 0, 0\right). \tag{9}$$

The sum of (7), (8) and (9) with coefficients 9/7, 3/7 and 6/7 is the following inequality:

$$\frac{9}{7}\alpha(1-\alpha) \leqslant q_\alpha \left(0, \frac{1}{2}, \frac{4}{7}, \frac{1}{2}, \frac{9}{14}, \frac{5}{14}, 0, 0, \frac{1}{7}, \frac{3}{14}, 0\right). \tag{10}$$

Since q_α is positive, we infer from (10) that

$$\frac{9}{7}\alpha(1-\alpha) \leqslant q_\alpha \left(0, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1\right) = q_\alpha \left(\overline{P_3} + K_3\right).$$

□

4 Improved bound

This section is devoted to the proof of Theorem 4. We equivalently prove the following.

Theorem 6. *For every $\alpha \in [0, 2/9]$, it holds that $q_\alpha\left(\overline{P_3} + K_3\right) \geqslant \frac{3}{4}\alpha(3 - \sqrt{8\alpha+1})$.*

Proof. Let $\beta := q_\alpha(K_2)$. Note that $\beta \in [\alpha, 1/2]$. We first derive two equalities using the fact that q_α is a homomorphism from $\mathcal{A}$ to $\mathbf{R}$. The first equation is a trivial corollary of this fact.

$$1 = q_\alpha(K_1) = q_\alpha(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1). \tag{11}$$

The choice of β implies that

$$\beta = q_\alpha\left(K_2\right) = q_\alpha\left(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6}, 1\right). \tag{12}$$

The next equality is little bit more tricky. We use that $q_\alpha\left(K_2\right) - \beta = 0$.

$$0 = (q_\alpha\left(K_2\right) - \beta) q_\alpha\left(\overline{K}_2\right) = q_\alpha\left(K_2 \times \overline{K}_2 - \beta \overline{K}_2\right). \tag{13}$$

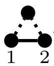
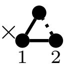
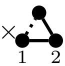
Again, we express (13) in terms of the four-vertex graphs:

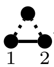
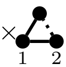
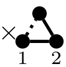
$$0 = q_\alpha\left(-\beta, \frac{1-5\beta}{6}, \frac{-2\beta}{3}, \frac{1-2\beta}{3}, \frac{1-\beta}{2}, \frac{1-3\beta}{6}, \frac{1-\beta}{2}, \frac{-\beta}{3}, \frac{1-\beta}{3}, \frac{1-\beta}{6}, 0\right). \tag{14}$$

The next inequality is the inequality (9) established in the proof of Theorem 5. We copy the inequality to ease the reading.

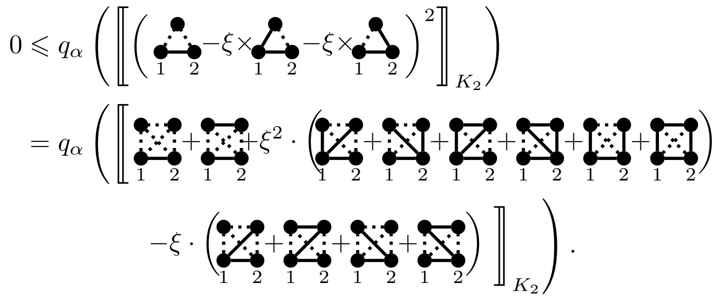
$$0 \leqslant q_\alpha\left(0, 0, 0, \frac{1}{6}, 0, \frac{1}{3}, -\frac{1}{2}, 0, -\frac{1}{3}, 0, 0\right). \tag{15}$$

The final inequality is obtained by considering random homomorphisms $q_\alpha^{K_2}$. Since $q_\alpha^{K_2}$ is a homomorphism, it holds for every choice of $q_\alpha^{K_2}$ and every $\xi \in \mathbf{R}$ that

$0 \leqslant q_\alpha^{K_2}\Bigg($  $-\xi\times$  $-\xi\times$  $\Bigg)^2$

$= q_\alpha^{K_2}\Bigg(\Bigg($  $-\xi\times$  $-\xi\times$  $\Bigg)^2\Bigg).$ (16)

Hence,



(17)

Evaluating the operator $[\![\cdot]\!]_{K_2}$ yields the following inequality.

$$0 \leqslant q_\alpha \left(0, \frac{1}{6}, \frac{1}{3}, \frac{-\xi}{3}, 0, \frac{\xi^2 - 2\xi}{6}, \frac{\xi^2}{2}, \frac{2\xi^2}{3}, \frac{\xi^2}{6}, 0, 0\right). \tag{18}$$

Now, let us sum the equations and inequalities (11), (12), (14), (15) and (18) with coefficients $\frac{3\beta}{\sqrt{1+8\beta}}$, $\frac{3}{4} \cdot \left(3 - \frac{5+8\beta}{\sqrt{1+8\beta}}\right)$, $\frac{3}{\sqrt{1+8\beta}}$, $\frac{3}{\sqrt{1+8\beta}}$ and $\frac{3}{4} \cdot \left(1 + \frac{1+4\beta}{\sqrt{1+8\beta}}\right)$, respectively, and substitute $\xi = \frac{\sqrt{1+8\beta}-1}{2\beta} - 1$. Note that the coefficients for the inequalities (15) and (18) are non-negative. So, we eventually deduce that

$$\frac{3}{4}\beta(3 - \sqrt{8\beta+1}) \leqslant q_\alpha \Bigg(0, \frac{1}{2}, 1 - \frac{1}{\sqrt{1+8\beta}}, \frac{1}{2}, \frac{9}{8} - \frac{3+12\beta}{8\sqrt{1+8\beta}}, \frac{1}{2},$$
$$0, 0, \frac{9}{8} - \frac{15+12\beta}{8\sqrt{8\beta+1}}, \frac{15}{8} - \frac{21+20\beta}{8\sqrt{1+8\beta}}, \frac{9}{4} - \frac{15+12\beta}{4\sqrt{1+8\beta}}\Bigg). \tag{19}$$

Finally, since q_α is positive, we derive from (19) that

$$\frac{3}{4}\beta(3 - \sqrt{8\beta+1}) \leqslant q_\alpha \left(0, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1\right) = q_\alpha \left(\overline{P_3} + K_3\right). \tag{20}$$

Observe that the function $x \mapsto \frac{3}{4}x(3-\sqrt{8x+1})$ is increasing on the interval $[0\,,2/9]$ and that

$$\frac{3}{4}x(3-\sqrt{8x+1}) \geqslant 2/9 = \frac{3}{4}\cdot\frac{2}{9}\left(3-\sqrt{8\cdot 2/9+1}\right)$$

for $x \in [2/9\,,1/2]$. Hence, the left hand side of (20) is at least $\frac{3}{4}\alpha(3-\sqrt{8\alpha+1})$ for $\alpha \in [0\,,2/9]$ as asserted in the statement of the theorem. □

5 Conclusion

Using more sophisticated methods, we have been able to further improve the bounds on $\varphi_2(\alpha)$. However, the proof becomes extremely complicated and since we have not been able to prove that

$$\varphi_2(\alpha) = \frac{3\alpha(1+\sqrt{1-4\alpha})}{4},$$

which is the bound given by the best known example, we have decided not to further pursue our work in this direction. To show the limits of our current approach, let us mention that Theorem 4 asserts that $\varphi_2(1/12) \geqslant 0.10681$ and we can push the bound to $\varphi_2(1/12) \geqslant 0.11099$; the simple bound is 0.08333 and the expected bound is 0.11353 for this value.

We have also attempted together with Andrzej Grzesik to apply this method for improving bounds on φ_3. Though we have been able to obtain some improvements, e.g., we can show that $\varphi_3(1/20) \geqslant 0.05183$, the level of technicality of the argument seems to be too large for us to be able to report on our findings in an accessible way at this point.

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