Classes of graphs with small rank decompositions are χ-bounded

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Abstract

A class of graphs $\mathcal{G}$ is χ-bounded if the chromatic number of graphs in $\mathcal{G}$ is bounded by a function of the clique number. We show that if a class $\mathcal{G}$ is χ-bounded, then every class of graphs admitting a decomposition along cuts of small rank to graphs from $\mathcal{G}$ is χ-bounded. As a corollary, we obtain that every class of graphs with bounded rank-width (or equivalently, clique-width) is χ-bounded.

1 Introduction

For a graph G and an integer k, a *proper k-coloring* of G is a function $f : V(G) \to \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ for every edge $uv \in E(G)$. The *chromatic number* $\chi(G)$ of G is the smallest k such that

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G has a proper k-coloring. The chromatic number is one of the most studied graph parameters, and although determining it precisely is NP-complete [3], a number of interesting bounds and connections to other graph parameters is known. A natural lower bound for the chromatic number is given by the *clique number* $\omega(G)$ which is the size of the largest complete subgraph of G. At first, it might be natural to believe that there could exist an upper bound on the chromatic number in the terms of the clique number. However, that is far from the truth. Erdős [2] showed that for any g there exist graphs of arbitrarily large chromatic number that do not contain cycles of length at most g.

Therefore, the graph classes where the chromatic and clique numbers are tied together are of an interest. The most famous example is the class of *perfect graphs* which are the graphs G such that $\chi(H) = \omega(H)$ for every induced subgraph H of G. The Strong Perfect Graph Theorem asserts that this class can be alternatively characterized as the class of graphs G such that neither G nor its complement contains an induced odd cycle of length at least 5 (Chudnovsky et al. [1]). The class of perfect graphs includes line graphs of bipartite graphs, chordal graphs, comparability graphs and others.

For many graph classes, the connection between χ and ω is not straightforward. For example, if G is a circle-arc graph (an intersection graph of arcs of a circle), then $\chi(G) \leq 2\omega(G)$. This leads to the following definition. We say that a class $\mathcal{G}$ of graphs is *χ-bounded* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$. The examples of χ-bounded graph classes include the circle-arc graphs, the circle graphs [6], line-graphs [8], and graphs avoiding any fixed tree T of radius at most two as an induced subgraph [5]. Also note that any class of graphs with bounded chromatic number is χ-bounded.

Our work is motivated by the following conjecture of Geelen:

Conjecture 1. *For every graph H, the class of graphs without a vertex-minor isomorphic to H is χ-bounded.*

Recall that a graph H' is a *vertex-minor* of a graph H if H'

can be obtained by a series of vertex removals and neighborhood complementations. The *neighborhood complementation* with respect to a vertex v of a graph G is the following operation: if two neighbors of v are joined by an edge, delete this edge from G, and if they are not joined, add an edge joining them.

A possible approach to Conjecture 1 could consist of proving that every graph without a vertex-minor isomorphic to H admits a decomposition to well-behaved pieces along simple-structured cuts. Such an approach has led to proofs of several deep results in structural graph theory. Examples include the graph minor structure theorem, where the graphs avoiding a minor of some fixed graph are obtained by joining pieces that are (almost) embedded in surfaces of small genus along small vertex cuts [7], and the proof of the Strong Perfect Graph Theorem [1].

In this paper, we introduce a new (somewhat technical) way of decomposing graphs along cuts of small rank. This kind of decomposition will allow us to prove two special cases of Conjecture 1.

Theorem 1. *For any k, the class of graphs with rank-width at most k is χ-bounded.*

Theorem 2. *The class of graphs without a vertex-minor isomorphic to the wheel W_5 is χ-bounded.*

We derive Theorem 2 from a structural characterization of graphs avoiding W_5 by Geelen [4] and another corollary of our main result. Recall that a 1-join of two graphs G_1 and G_2 with two distinguished vertices v_1 and v_2, respectively, is the graph obtained by deleting the vertex v_i from G_i, $i = 1, 2$, and adding an edge between every neighbor of v_1 in G_1 and every neighbor of v_2 in G_2. The corollary we use to derive Theorem 2 is the following result.

Theorem 3. *Let $\mathcal{G}$ be a χ-bounded class of graphs closed under taking induced subgraphs. If $\mathcal{G}'$ is the class of graphs that can be obtained from graphs of $\mathcal{G}$ by repeated applications of 1-joins, then the class $\mathcal{G}'$ is also χ-bounded.*

In the next section, we formally introduce the decompositions we study and derive Theorems 1–3 from our main result (Theorem 4). Theorem 4 is then proven in Section 3.

2 Decomposition along small cuts

If we want to allow the existence of cliques of arbitrary order, we cannot restrict the size of the cuts along those we split. On the other hand, such cuts should not be completely arbitrary and thus it is natural to restrict their complexity. This leads to the following definition. For a graph H and a proper subset W of vertices of H, the *matrix of the cut* $(W, V(H) \setminus W)$ is the $|W| \times |V(H) \setminus W|$ matrix M indexed by W and $V(H) \setminus W$ such that M_{uv} is 1 if $uv \in E(H)$ and 0 otherwise. The *rank* of the cut is the rank of its matrix over F_2. The *diversity* of the cut is the size of the largest set S such that $S \subseteq W$ or $S \subseteq V(H) \setminus W$ and the vertices of W have mutually distinct neighborhoods in the other side of the cut. In other words, the diversity of the cut is the maximum number of different rows or columns of the its matrix. Note that if the rank of the cut is r, then its diversity is at most 2^r, and, conversely, if the diversity is d, then its rank is at most d.

A natural way of decomposing a graph is assigning its vertices to nodes of a tree. Formally, a *decomposition* of a graph G is a tree T with a mapping $\tau : V(G) \to V(T)$. Each edge e of the tree T naturally defines a cut in the graph G with sides being the preimages of the vertex sets of the two trees obtained from T by removing e. The *rank* of a decomposition is the maximum of the ranks of the cuts induced by its edges. Analogously, we define the *diversity* of a decomposition.

Every graph admits a decomposition of rank one with T being a star, so restricting the rank of decompositions is too weak. One way to circumvent this is to restrict the structure of the decomposition. For example, if we require that all inner nodes of T have degree three and that the vertices of G are injectively mapped by τ to the leaves of T, the rank of the smallest such decomposition is the *rank-width* of G, a well-studied parameter in structural graph theory. In this paper, we proceed in a different (more general) way.

If a tree T with a mapping τ is a decomposition of a graph G and v is a node of T, then $G_{T,v}$ is the spanning subgraph of G such that two vertices u and u' of G are adjacent in $G_{T,v}$ if they are adjacent in G and the node v lies on the unique path between $\tau(u)$ and $\tau(u')$ in T (possibly, v can be $\tau(u)$ or $\tau(u')$). In other words, $G_{T,v}$ is the spanning subgraph of G where we remove edges between vertices u and u' such that $\tau(u)$ and $\tau(u')$ lies in the same component of $T \setminus v$. For a class $\mathcal{G}$ of graphs, we say that the decomposition T is *$\mathcal{G}$-bounded* if the graph $G_{T,v}$ belongs to $\mathcal{G}$ for every $v \in V(T)$.

Our main result is the following theorem.

Theorem 4. *Let $\mathcal{G}$ be a χ-bounded class of graphs closed under taking induced subgraphs and r an integer. The class of graphs admitting a $\mathcal{G}$-bounded decomposition with rank at most r is χ-bounded.*

We would like to note that the definition of $\mathcal{G}$-bounded decompositions can also be extended as follows. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two classes of graphs. We can require that the subgraphs of G induced by $\tau^{-1}(v)$, $v \in V(T)$, belong to $\mathcal{G}_1$ and the subgraphs $G_{T,v} \setminus \tau^{-1}(v)$ to $\mathcal{G}_2$. So, the class $\mathcal{G}_1$ controls the complexity of subgraphs induced by preimages of individual nodes and the class $\mathcal{G}_2$ controls the mutual interaction between different pieces of the decomposition. However, if both $\mathcal{G}_1$ and $\mathcal{G}_2$ are χ-bounded and closed under taking induced subgraphs, this does not lead to a more general concept. Indeed, let $\mathcal{G}$ be the class of graphs G that admit a vertex partition V_1 and V_2 such that $G[V_i] \in \mathcal{G}_i$ for $i = 1, 2$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are χ-bounded and closed under taking induced subgraphs, so is $\mathcal{G}$, and any decomposition with respect to $\mathcal{G}_1$ and $\mathcal{G}_2$ as defined in this paragraph is $\mathcal{G}$-bounded.

We now derive Theorems 1–3 from Theorem 4. The decompositions appearing in the definition of the rank-width of a graph, which is given earlier in this section, are $\mathcal{G}$-bounded where $\mathcal{G}$ is the class of

3-colorable graphs. So, an application of Theorem 4 for this class $\mathcal{G}$ yields Theorem 1.

The proof of Theorem 3 is more complicated.

Proof of Theorem 3. Let G be a graph contained in $\mathcal{G}'$. By the definition of $\mathcal{G}'$, there exists a tree T with nodes $v_1, \ldots, v_n$ and graphs $G_1, \ldots, G_n$ from $\mathcal{G}$ such that

- for every edge $v_i v_j$ of T, the graphs G_i and G_j contain distinguished vertices $w_{i,j} \in V(G_i)$ and $w_{j,i} \in V(G_j)$, respectively, and these vertices are different fordifferent choices of $v_i v_j$, and

- the graph G is obtained from the graphs $G_1, \ldots, G_n$ by 1-joins with respect to vertices $w_{i,j}$ and $w_{j,i}$, $v_i v_j \in E(T)$.

Note that the order of 1-joins does not affect the result of the procedure.

Let $\mathcal{G}''$ be the class of graphs that can be obtained from $\mathcal{G}$ by adding isolated vertices and blowing up some vertices to independent sets, i.e., replacing a vertex w with and independent set I and joining each vertex of I to the neighbors of w. Since $\mathcal{G}$ is χ-bounded and closed under taking induced subgraphs, so is $\mathcal{G}''$. Since the tree T with mapping τ that maps a vertex $u \in V(G)$ to the node v_i such that $u \in V(G_i)$ is a $\mathcal{G}''$-bounded decomposition of G and its rank is at most one, Theorem 3 now follows from Theorem 4. $\square$

To derive Theorem 2 from Theorem 3, we need the following result of Geelen [4]. Observe that since the class of circle graphs is χ-bounded [6] and all other basic graphs appearing in Theorem 5 have at most 8 vertices, Theorem 2 directly follows from Theorem 3.

Theorem 5 (Geelen [4], Theorem 5.14)**.** *If G is a connected graph without a vertex-minor isomorphic to W_5, then one of the following holds:*

- *G is a circle graph, or*

- *G is can be obtained from a graph isomorphic to W_7, the cube C or the graph C^-, that is the cube with one vertex removed, by a sequence of neighborhood complementations, or*
- *there exist connected graphs G_1 and G_2 without a vertex-minor isomorphic to W_5 that have fewer vertices than G, such that G is a 1-join of G_1 and G_2.*

3 Proof of Main Theorem

In this section, we present the proof of Theorem 4. We start with a lemma.

Lemma 6. *Let d and k be two integers and G a connected graph with at least two vertices. If G has a decomposition formed by a tree T and a mapping τ such that the diversity of the decomposition is at most d and $\chi(G_{T,v}) \leq k$ for every node v of T, then there exists a (not necessarily proper) coloring of the vertices of G by at most $d(k+1)$ colors such that $\omega(G[\varphi^{-1}(c)]) < \omega(G)$ for every color c.*

Proof. We can assume (without loss of generality) that T has a leaf v_1 with $\tau^{-1}(v_1) = \emptyset$ and root T at this leaf. Fix a sequence $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n$ of subtrees of T such that T_1 is the subtree formed by v_1 solely, $T_n = T$ and $|V(T_i) \setminus V(T_{i-1})| = 1$ for every $i = 2, \ldots, n$. Let v_i be the vertex of T_i not contained in T_{i-1}. For an edge $uu' \in E(G)$, the *origin* of uu' is the nearest common ancestor of $\tau(u)$ and $\tau(u')$ in T.

For a node v of T, let T_v be the subtree of T rooted at v and let V_v be the union $\bigcup_{v' \in V(T_v)} \tau^{-1}(v')$. Since the diversity of the decomposition given by T and τ is at most d, there exists a partition $V_v^0, \ldots, V_v^d$ of V_v such that the vertices of V_v^0 have no neighbors outside V_v and two vertices of V_v belong to the same V_v^j if and only if they have the same neighbors outside of V_v.

The set of colors used by the constructed coloring will be $C_0 = \{1, \ldots, d(k+1)\}$. We will construct partial (not necessarily proper) colorings $\varphi_1, \varphi_2, \ldots, \varphi_n$ of G which we use the colors C_0 such that

1. φ_i extends φ_{i-1} for $i = 2, \ldots, m$,

2. φ_i assigns colors to all vertices incident with edges whose origin belongs to T_i, in particular, to all vertices of $\tau^{-1}(V(T_i))$,

3. the vertices of $V^j_{v_{i'}}$ for $i' > i$ and $j = 0, \ldots, d$ are either all colored with the same color or none of them is colored by φ_i, $i = 1, \ldots, n$, and

4. if $\varphi_i(u) = \varphi_i(u')$ for an edge uu' of G and $i \in \{1, \ldots, n\}$, then there exists i', $2 \le i' \le i$, a child v' of $v_{i'}$, and $j \in \{1, \ldots, d\}$ such that both u and u' are not colored by $\varphi_{i'-1}$, they are colored by $\varphi_{i'}$ and they both belong to $V^j_{v'}$.

Since v_1 is the origin of no edge, we can choose φ_1 as the empty coloring. Suppose that φ_{i-1} was already defined and let us describe the construction of φ_i from φ_{i-1}. Note that the vertices of V_{v_i} that are not colored by φ_{i-1} are exactly those of $V^0_{v_i}$. Furthermore, the restriction of φ_{i-1} to V_{v_i} uses at most d colors (one for each of $V^1_{v_i}$, $\ldots$, $V^d_{v_i}$). Let C be the set of these colors.

By the assumption of the lemma, there exists a proper coloring ψ_1 of the graph G_{T,v_i} using k colors. Choose ψ_1 such that all twins, i.e., vertices with the same set of neighbors, receive the same color. Let W be the vertices of $V^0_{v_i}$ that have a non-zero degree in G_{T,v_i} or are mapped by τ to v_i. Let ψ_2 be a (not necessarily proper) coloring of the vertices of W by colors $1, \ldots, d$ defined as follows: if $\tau(w) = v_i$, the vertex w is assigned the color 1. For $w \in W \setminus \tau^{-1}(v_i)$, let v' be the child of v_i such that $w \in V_{v'}$. Set $\psi_2(w)$ to be the index j such that $w \in V^j_{v'}$ (note that $j > 0$). Finally, set $\psi(w) = (\psi_1(w), \psi_2(w))$; ψ is a (not necessarily proper) coloring of the vertices of W with $d \cdot k$ colors.

The coloring φ_i is an extension of φ_{i-1} to the vertices of W such that φ_i uses the (at least) $d \cdot k$ colors of $C_0 \setminus C$ and two vertices of W get the same color if and only if they are assigned the same color by ψ.

We now verify that φ_i has the properties stated earlier.

1. This follows directly from the definition of φ_i.

2. If the origin of an edge uu' of G belongs to T_i, then it either belongs to T_{i-1} or it is v_i. In the former case, u and u' are already colored by φ_{i-1}. In the latter case, they both belong to W and are colored by φ_i.

3. If any vertex of $V^j_{v_{i'}}$ is colored by φ_{i-1}, then all of them are colored by φ_{i-1} and they have the same color. So, suppose that none of the vertices of $V^j_{v_{i'}}$ is colored by φ_{i-1}. Observe that all vertices of $V^j_{v_{i'}}$ are twins in G_{T,v_i}. So, if one of them belongs to W, then all of them do. Also, they are assigned the same color by ψ_1. Let v' be the child of v_i such that $v_{i'}$ belongs to $T_{v'}$. Since the vertices of $V^j_{v_{i'}}$ have the same neighbors outside $V_{v'}$, they all belong to the same set $V^{j'}_{v'}$. Consequently, they are assigned the same color by ψ_2. So, they have the same color assigned by ψ and thus by φ_i, too.

4. Suppose that $\varphi_i(u) = \varphi_i(u')$ and uu' is an edge of G. If both u and u' are colored by φ_{i-1}, the property follows from the properties of φ_{i-1}. By symmetry, we can now assume that u is not colored by φ_{i-1}. Since u is not colored, u' must belong to V_{v_i}. If u' were colored by φ_{i-1}, it would belong to one of the sets $V^1_{v_i}, \ldots, V^d_{v_i}$. So, it would hold that $\varphi_i(u') \in C$ but $\varphi_i(u) \in C_0 \setminus C$ which is impossible. We conclude that neither u nor u' is colored by φ_{i-1}. Since they are colored by φ_i, they both belong to W, and, since φ_i assigns them the same color, it holds that $\psi(u) = \psi(u')$. The definition of ψ_1 implies that both u and u' belong to the same $V_{v'}$ for a child v' of v_i (otherwise, the edge uu' would be contained in G_{T,v_i}). Finally, the definition of ψ_2 implies that they both belong to the same $V^j_{v'}$ for $j > 0$.

We claim that φ_n is the sought coloring. The number of colors it uses does not exceed $d(k+1)$. Assume that G contains a monochromatic maximum clique K. The fourth property of the coloring φ_n

implies that all vertices of K belong to V_v^j for some node v of T and $j > 0$. In particular, they have a common neighbor which is impossible since K is a maximum clique. $\square$

The proof of the main theorem now follows.

Proof of Theorem 4. Let f be the function witnessing that $\mathcal{G}$ is χ-bounded and let $f'(s) = 2^{rs}\prod_{i=2}^{s}(f(s)+1)$. We show that $\chi(G) \leq f'(\omega(G))$ for any graph G admitting a $\mathcal{G}$-bounded decomposition with rank at most r by induction on the clique number $\omega(G)$.

If $\omega(G) = 1$, then $\chi(G) = 1 \leq f'(1)$. Assume that $\omega(G) > 1$. By Lemma 6, the vertices of G can be colored with $2^r(f(\omega(G))+1)$ colors in such a way that no clique of G of size $\omega(G)$ is monochromatic. Fix such a coloring φ. Consider the subgraphs of G induced by the color classes of φ. Since each of these subgraphs admits a $\mathcal{G}$-bounded decomposition with rank at most r, it has a proper coloring with $f'(\omega(G)-1)$ colors. Coloring each vertex by a pair consisting of the color assigned by φ and the color assigned by the coloring of the corresponding color class of G yields a proper coloring of G with $2^r(f(\omega(G))+1)f'(\omega(G)-1) = f(\omega(G))$ colors. $\square$

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