Bounds on eigenvalues of complex interval matrices

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Abstract

We present a cheap and tight formula for bounding real and imaginary parts of eigenvalues of (complex) interval matrices. It generalizes and improves the results by Rohn (1998) and Hertz (2009). The main idea behind is to reduce the problem to enclosing eigenvalues of symmetric interval matrices, for which diverse methods can be utilized.

The result helps in analysing stability of uncertain dynamical systems since the formula gives sufficient conditions for testing Schur and Hurwitz stability of interval matrices. It may also serve as a starting point for some iteration methods.

Keywords: *Interval matrix, interval analysis, eigenvalue, eigenvalue bounds.*

1 Introduction

Stability of dynamical systems attracts attention of control community for several decades. Discrete dynamical systems lead to a Schur stability, and continuous systems lead to a Hurwitz stability of the

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system matrices. A matrix A is Schur stable if its spectral radius is less than 1, and A is Hurwitz if real parts of all eigenvalues are negative.

There are intrinsic uncertainties when solving practical problems. Uncertainties are modelled in diverse ways, but interval analysis naturally handles the best and worst cases of continuous domains of parameters. The aim od this paper is to derive cheap but sharp bounds on eigenvalues of complex interval matrices.

An interval matrix is defined as a family of matrices

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}$, $\overline{A} \in \mathbb{R}^{n\times n}$, $\underline{A} \le \overline{A}$, are given matrices, and the inequality is considered element-wise. The midpoint and the radius of $\boldsymbol{A}$ are denoted respectively by

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $n \times n$ interval matrices is denoted by $\mathbb{IR}^{n\times n}$. A complex interval matrix as a family $\boldsymbol{A} + i\boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are interval matrices of order n. The eigenvalue set $\Lambda(\boldsymbol{A} + i\boldsymbol{B})$ corresponding to $\boldsymbol{A}+i\boldsymbol{B}$ is defined as the set of all eigenvalues over all $A+iB \in \boldsymbol{A}+i\boldsymbol{B}$, that is,

$$\begin{aligned}\Lambda(\boldsymbol{A} + i\boldsymbol{B}) = \{\lambda + i\mu \mid \exists A \in \boldsymbol{A}\ \exists B \in \boldsymbol{B}\ \exists x + iy \ne 0 :& \\ (A + iB)(x + iy) = (\lambda + i\mu)(x + iy)\}&.\end{aligned}$$

Deif [3] presents a description of the eigenvalue sets and their exact bounds, however, it is valid only under some assumptions on sign pattern invariance of eigenvectors, which are not so easy to verify. In praxis, it mostly suffices to calculate a fast computable enclosures (supersets) of eigenvalue sets. Kolev and Petrakieva [11] develop an enclosure for real parts eigenvalues by solving nonlinear system of equations; an exact bound can be achieved under some monotonicity assumptions. Mayer [12] proposes an enclosure method for eigenvalues of real and complex interval matrices based on Taylor expansion. A cheap formula for an enclosure is in Rohn [17]. An estimation on eigenvalues based on perturbation theory appears in Ahn et al. [2].

For Hurwitz stability checking, Franze et al. [4] present a sufficient condition by using a Gershgorin-type theorem. Xiao and Unbehauen [21] show that Schur/Hurwitz stability checking can be reduced to checking only exposed faced of an interval matrix. Further, Rohn [15] proved that Hurwitz stability can be reduced to inspecting 2^{2n-1} special vertex matrices provided that each matrix in $\boldsymbol{A}$ has real eigenvalues only. Stability analysis based on Lyapunov equation was studied in [1, 14, 20].

A special subclass of interval matrices are symmetric interval matrices. For an interval matrix $\boldsymbol{A}$, the corresponding symmetric interval matrix $\boldsymbol{A}^S$ is defined as a family of all symmetric matrices in $\boldsymbol{A}$, that is,

$$\boldsymbol{A}^S = \{A \in \boldsymbol{A} \mid A = A^T\}.$$

A real symmetric matrix $A \in \mathbb{R}^{n \times n}$ has n real eigenvalues; we can assume that they are sorted in a non-increasing order as follows

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Extending the notation for symmetric interval matrices, we introduce

$$\boldsymbol{\lambda}_i(\boldsymbol{A}^S) = [\underline{\lambda}_i(\boldsymbol{A}^S), \overline{\lambda}_i(\boldsymbol{A}^S)] := \big\{\lambda_i(A) \mid A \in \boldsymbol{A}^S\big\}, \quad i = 1, \ldots, n,$$

the eigenvalue sets of a symmetric interval matrix $\boldsymbol{A}^S$. The eigenvalue sets represent n compact intervals, which are either disjoint or may overlap [8].

Even though the main focus of this paper is on bounding eigenvalues of general (complex) interval matrices, we review some methods for symmetric interval matrices since our main result reduces computation of the general case to the symmetric one. Various bounding methods are discussed in Hladík et al. [8]. For checking Schur/Hurwitz stability a symmetric interval matrix, [16] proposed a branch & bound algorithm. Other related results are found in [6, 7, 9, 18].

The following simple but efficient bounds are due to Rohn [18].

Theorem 1 (Rohn, 2005)**.** *For each $i \in \{1, \dots, n\}$ one has*

$$\boldsymbol{\lambda}_i(\boldsymbol{A}^S) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)].$$

As shown by Hertz [6, 7], the extremal eigenvalue limits $\overline{\lambda}_1(\boldsymbol{A}^S)$ and $\underline{\lambda}_n(\boldsymbol{A}^S)$ can be computed exactly by inspecting 2^{n-1} special vertex matrices.

Theorem 2 (Hertz, 1992)**.** *Define $Z := \{1\} \times \{\pm 1\}^{n-1} = \{(1, \pm 1, \dots, \pm 1)\}$ and for a $z \in Z$ define $A_z, A'_z \in \boldsymbol{A}^S$ in this way:*

$$(a_z)_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } z_i = z_j, \\ \underline{a}_{ij} & \text{if } z_i \neq z_j, \end{cases}, \quad (a'_z)_{ij} = \begin{cases} \underline{a}_{ij} & \text{if } z_i = z_j, \\ \overline{a}_{ij} & \text{if } z_i \neq z_j. \end{cases}$$

Then

$$\overline{\lambda}_1(\boldsymbol{A}^S) = \max_{z \in Z} \lambda_1(A_z), \quad \underline{\lambda}_n(\boldsymbol{A}^S) = \min_{z \in Z} \lambda_n(A'_z).$$

The main focus of this paper is on bounding complex eigenvalues of interval matrices. One of the basic bounds is the following formula by Rohn [17].

Theorem 3 (Rohn, 1998)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then for each eigen-*

value $\lambda + i\mu \in \Lambda(\boldsymbol{A})$ *we have*

$$\begin{aligned}
\lambda &\le \lambda_1\left(\frac{1}{2}(A_c + A_c^T)\right) + \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),\\
\lambda &\ge \lambda_n\left(\frac{1}{2}(A_c + A_c^T)\right) - \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),\\
\mu &\le \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T)\\ \frac{1}{2}(A_c^T - A_c) & 0\end{pmatrix}\\
&\quad + \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T)\\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0\end{pmatrix},\\
\mu &\ge \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T)\\ \frac{1}{2}(A_c^T - A_c) & 0\end{pmatrix}\\
&\quad - \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T)\\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0\end{pmatrix}.
\end{aligned}$$

The Rohn's result was generalized to complex interval matrices by Hertz [7].

Theorem 4 (Hertz, 2009)**.** *Let* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$*. Then for each* $\lambda +$

$i\mu \in \Lambda(\boldsymbol{A} + i\boldsymbol{B})$ *we have*

$$\begin{aligned}
\lambda \leq\ & \lambda_1\left(\frac{1}{2}(A_c + A_c^T)\right) + \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right) \\
& + \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & 0 \end{pmatrix} \\
& + \rho\begin{pmatrix} 0 & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & 0 \end{pmatrix}, \\
\lambda \geq\ & \lambda_n\left(\frac{1}{2}(A_c + A_c^T)\right) - \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right) \\
& + \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & 0 \end{pmatrix} \\
& - \rho\begin{pmatrix} 0 & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & 0 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mu \leq\ & \lambda_1\left(\frac{1}{2}(B_c + B_c^T)\right) + \rho\left(\frac{1}{2}(B_\Delta + B_\Delta^T)\right) \\
& + \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} \\
& + \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix}, \\
\mu \geq\ & \lambda_n\left(\frac{1}{2}(B_c + B_c^T)\right) - \rho\left(\frac{1}{2}(B_\Delta + B_\Delta^T)\right) \\
& + \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} \\
& - \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix}.
\end{aligned}$$

2 Main result

Now we present our main result. It generalizes the results of Rohn and Hertz, and enables to derive sharper bounds.

Theorem 5. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$. Then for each eigenvalue $\lambda + i\mu \in \Lambda(\boldsymbol{A} + i\boldsymbol{B})$ we have*

$$\underline{\lambda}_n \begin{pmatrix} \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) & \frac{1}{2}(\boldsymbol{B}^T-\boldsymbol{B}) \\ \frac{1}{2}(\boldsymbol{B}-\boldsymbol{B}^T) & \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) \end{pmatrix}^S \leq \lambda \leq \overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) & \frac{1}{2}(\boldsymbol{B}^T-\boldsymbol{B}) \\ \frac{1}{2}(\boldsymbol{B}-\boldsymbol{B}^T) & \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) \end{pmatrix}^S,$$

$$\underline{\lambda}_n \begin{pmatrix} \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T) & \frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^T) \\ \frac{1}{2}(\boldsymbol{A}^T-\boldsymbol{A}) & \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T) \end{pmatrix}^S \leq \mu \leq \overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T) & \frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^T) \\ \frac{1}{2}(\boldsymbol{A}^T-\boldsymbol{A}) & \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T) \end{pmatrix}^S.$$

Proof. Let $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, let $\lambda + i\mu$ be an eigenvalue of $A + iB$ and $x + iy$ the corresponding eigenvector. Suppose that the eigenvector is normalized such that $1 = \|x + iy\|_2 = x^T x + y^T y$. Then we have $(A + iB)(x + iy) = (\lambda + i\mu)(x + iy)$, or by comparing the real and the imaginary parts separately

$$\begin{aligned} Ax - By &= \lambda x - \mu y, \\ Bx + Ay &= \lambda y + \mu x. \end{aligned}$$

Multiplying the first equation by x^T and the second by y^T, and summing up we obtain

$$x^T Ax - x^T By + y^T Bx + y^T Ay = \lambda(x^T x + y^T y) = \lambda.$$

Now, from the Courant–Fischer theorem [5, 10, 13] it follows

$$
\begin{aligned}
&\overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) & \frac{1}{2}(\boldsymbol{B}^T-\boldsymbol{B}) \\ \frac{1}{2}(\boldsymbol{B}-\boldsymbol{B}^T) & \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) \end{pmatrix}^S \\
&\geq \lambda_1 \begin{pmatrix} \frac{1}{2}(A+A^T) & \frac{1}{2}(B^T-B) \\ \frac{1}{2}(B-B^T) & \frac{1}{2}(A+A^T) \end{pmatrix} \\
&= \max_{\|(u,v)\|=1} (u^T, v^T) \begin{pmatrix} \frac{1}{2}(A+A^T) & \frac{1}{2}(B^T-B) \\ \frac{1}{2}(B-B^T) & \frac{1}{2}(A+A^T) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&\geq (x^T, y^T) \begin{pmatrix} \frac{1}{2}(A+A^T) & \frac{1}{2}(B^T-B) \\ \frac{1}{2}(B-B^T) & \frac{1}{2}(A+A^T) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= x^T \frac{1}{2}(A+A^T)x + x^T \frac{1}{2}(B^T-B)y \\
&\quad + y^T \frac{1}{2}(B-B^T)x + y^T \frac{1}{2}(A+A^T)y \\
&= x^T A x + y^T A y + x^T (B^T - B) y \\
&= \lambda.
\end{aligned}
$$

Similarly by obtain the lower bound on λ. To show the lower and the upper bound on μ we can proceed analogously, or we can realize that an imaginary part of an eigenvalue of an interval complex matrix $A + iB$ is equal to the real part of the eigenvalue of $-i(A + iB) = B - iA$. □

Theorem 5 reduces the problem of enclosing complex eigenvalues of interval complex matrices to bounding the maximal and minimal eigenvalues of symmetric interval matrices. Thus, any bound on extremal eigenvalues of symmetric interval matrices can be utilized. A discussion of various cheap and tight enclosing formulae was addressed in Hladík et al. [8]; see also a filtering method in Hladík et al. [9]. Employing the exponential Hertz's formula (Theorem 2) we obtain the tightest bounds that Theorem 5 can serve for. Applying the basic Rohn's bounds from Theorem 1 we get the following corollary.

Corollary 1. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$. Then for each $\lambda + i\mu \in \Lambda(\boldsymbol{A} + i\boldsymbol{B})$ we have*

$$\lambda \le \lambda_1 \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{2}(A_\Delta + A_\Delta^T) & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{pmatrix}, \tag{1}$$

$$\lambda \ge \lambda_n \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} - \rho \begin{pmatrix} \frac{1}{2}(A_\Delta + A_\Delta^T) & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{pmatrix},$$

$$\mu \le \lambda_1 \begin{pmatrix} \frac{1}{2}(B_c + B_c^T) & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & \frac{1}{2}(B_c + B_c^T) \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & \frac{1}{2}(B_\Delta + B_\Delta^T) \end{pmatrix},$$

$$\mu \ge \lambda_n \begin{pmatrix} \frac{1}{2}(B_c + B_c^T) & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & \frac{1}{2}(B_c + B_c^T) \end{pmatrix} - \rho \begin{pmatrix} \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & \frac{1}{2}(B_\Delta + B_\Delta^T) \end{pmatrix}.$$

For a real interval matrix $\boldsymbol{A}$ we obtain the same result as in Theorem 3. This is easy to see since by putting $B_c = B_\Delta = 0$ e.g. to (1) we get

$$\begin{aligned}
&\lambda_1 \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{2}(A_\Delta + A_\Delta^T) & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{pmatrix} \\
&\quad = \lambda_1 \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & 0 \\ 0 & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} \\
&\quad + \rho \begin{pmatrix} \frac{1}{2}(A_\Delta + A_\Delta^T) & 0 \\ 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{pmatrix} \\
&\quad = \lambda_1 \left(\frac{1}{2}(A_c + A_c^T) \right) + \rho \left(\frac{1}{2}(A_\Delta + A_\Delta^T) \right).
\end{aligned}$$

Thus, using tighter bounds for eigenvalues of symmetric interval matrices than that by Rohn we obtain tighter bounds on the desired complex eigenvalues.

For a complex interval matrix $\boldsymbol{A} + i\boldsymbol{B}$ we obtain a stronger result than the bounds by Hertz (Theorem 4). This is because

$$
\begin{aligned}
&\lambda_1 \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} \\
&\quad \leq \lambda_1 \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & 0 \\ 0 & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} \\
&\quad + \lambda_1 \begin{pmatrix} 0 & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & 0 \end{pmatrix} \\
&\quad = \lambda_1 \left(\frac{1}{2}(A_c + A_c^T) \right) + \lambda_1 \begin{pmatrix} 0 & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & 0 \end{pmatrix},
\end{aligned}
$$

and similarly for the other terms.

Another consequence is obtained for Hermitian matrices. Hermitian matrices are of the form $A + iB$, where A is symmetric and B skew-symmetric, ane they have always real eigenvalues. Consider an interval Hermitian matrix $\boldsymbol{A} + i\boldsymbol{B}$, where $\boldsymbol{A} = \boldsymbol{A}^T$ and $\boldsymbol{B} = -\boldsymbol{B}^T$. Even though the family of matrices $\boldsymbol{A}+i\boldsymbol{B}$ contains not only Hermitian matrices, we can bound any eigenvalue λ of a Hermitian matrix from $\boldsymbol{A} + i\boldsymbol{B}$ according to Theorem 4 as

$$
\underline{\lambda}_n \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}^S \leq \lambda \leq \overline{\lambda}_1 \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}^S .
$$

3 Examples

Example 1. Inspired by [19], consider a matrix in the form of $A(s) = (I - ss^T)B(s)$, where

$$
B(s) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} + i \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 2 \\ 4 & 2 & 0 \end{pmatrix} + 4i \begin{pmatrix} 0 & -s_1 - i\, s_2 & i\, s_3 \\ s_1 + i\, s_2 & 0 & -s_3 \\ -i\, s_3 & s_3 & 0 \end{pmatrix}.
$$

Let the vector of parameter s vary within $\boldsymbol{s} = ([0, 0.2], [0.9797, 1], 0)$.

The enclosure computed by the Hertz method (Theorem 4) is

$$
\lambda \in [-5.6732, 8.5134], \quad \mu \in [-7.4311, 11.8843].
$$

Our approach depends on the solver for the symmetric interval eigenvalue problem used. Employing the Rohn bounds (Theorem 1) we obtain

$$\lambda \in [-4.6546, 7.6146], \quad \mu \in [-5.7632, 10.3387],$$

which can be improved by the filtering method to

$$\lambda \in [-4.6546, 6.7421], \quad \mu \in [-5.4017, 9.8787].$$

The tightest enclosure is obtained by the exponential Hertz formula (Theorem 2)

$$\lambda \in [-4.5180, 6.3031], \quad \mu \in [-4.8237, 9.6227].$$

The results are illustrated in Figure 1. Tha basic extent is given by the complex Hertz enclosure, and the following enclosures are represented by nested rectangles. The eigenvalue set is approximated by the Monte Carlo method.

In a similar manner as in Example 1 we have considered some real interval matrices, too. Since the first two enclosures are equal in the real case, we display just the results of Theorem 5 by using the Rohn bounds, the filtering and the Hertz formula.

Example 2 (Figure 2)**.** Consider the interval matrix [14, Example 4.3]

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & -1 \\ 2 & [-1.399, -0.001] & 0 \\ 1 & 0.5 & -1 \end{pmatrix}$$

The enclosures obtained by using the Rohn bounds, the filtering and the Hertz formula are, respectively,

$$\begin{aligned} \lambda &\in [-1.9068, 0.9702], \quad \mu \in [-2.5191, 2.5191], \\ \lambda &\in [-1.6474, 0.5205], \quad \mu \in [-2.1934, 2.1934], \\ \lambda &\in [-1.6474, 0.5205], \quad \mu \in [-2.1112, 2.1112]. \end{aligned}$$

Example 3 (Figure 3)**.** Consider the interval matrix [21, Example 2]

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 & [-1,1] \\ 0 & -1 & [-1,1] \\ [-1,1] & [-1,1] & 0.1 \end{pmatrix}$$

The enclosures calculated are, respectively,

$$\begin{aligned} \lambda &\in [-2.4143, 1.5143], \quad \mu \in [-1.4143, 1.4143], \\ \lambda &\in [-2.0532, 1.1532], \quad \mu \in [-1.4143, 1.4143], \\ \lambda &\in [-1.9674, 1.0674], \quad \mu \in [-1.4143, 1.4143]. \end{aligned}$$

Example 4 (Figure 4)**.** Consider the interval matrix [20, Example 1]

$$\boldsymbol{A} = \begin{pmatrix} [-3,-2] & [4,5] & [4,6] & [-1,1.5] \\ [-4,-3] & [-4,-3] & [-4,-3] & [1,2] \\ [-5,-4] & [2,3] & [-5,-4] & [-1,0] \\ [-1,0.1] & [0,1] & [1,2] & [-4,2.5] \end{pmatrix}$$

We get the following enclosures

$$\begin{aligned} \lambda &\in [-8.8221, 3.4408], \quad \mu \in [-10.7497, 10.7497], \\ \lambda &\in [-7.4848, 3.3184], \quad \mu \in [-9.4224, 9.4224], \\ \lambda &\in [-7.3691, 3.2742], \quad \mu \in [-8.7948, 8.7948]. \end{aligned}$$

Conclusion

We presented a method for computing bounds on eigenvalues of complex interval matrices. The basic idea is to reduce the problem to the symmetric case, and utilize various methods for this case. Thus by using Rohn bounds or the filtering method we obtain very cheap and efficient formulae outperforming the complex Hertz method. By incorporating the symmetric Hertz method we get the tightest enclosure but on the account of the exponential computing time; moreover, the superiority to the filtering method is not significant.

References

[1] H.-S. Ahn, Y. Chen, and I. Podlubny. Robust stability test of a class of linear time-invariant interval fractional-order system using Lyapunov inequality. *Appl. Math. Comput.*, 187(1):27–34, 2007.

[2] H.-S. Ahn, K. L. Moore, and Y. Chen. Monotonic convergent iterative learning controller design based on interval model conversion. *IEEE Trans. Autom. Control*, 51(2):366–371, 2006.

[3] A. S. Deif. The interval eigenvalue problem. *Z. Angew. Math. Mech.*, 71(1):61–64, 1991.

[4] G. Franzè, L. Carotenuto, and A. Balestrino. New inclusion criterion for the stability of interval matrices. *IEE Proceedings - Control Theory and Applications*, 153(4):478–482, 2006.

[5] G. H. Golub and C. F. Van Loan. *Matrix computations. 3rd ed.* Johns Hopkins University Press, 1996.

[6] D. Hertz. The extreme eigenvalues and stability of real symmetric interval matrices. *IEEE Trans. Autom. Control*, 37(4):532–535, 1992.

[7] D. Hertz. Interval analysis: Eigenvalue bounds of interval matrices. In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of optimization*, pages 1689–1696. Springer, New York, 2009.

[8] M. Hladík, D. Daney, and E. Tsigaridas. Bounds on real eigenvalues and singular values of interval matrices. *SIAM J. Matrix Anal. Appl.*, 31(4):2116–2129, 2010.

[9] M. Hladík, D. Daney, and E. P. Tsigaridas. A filtering method for the interval eigenvalue problem. *Appl. Math. Comput.*, (12):5236–5242, 2011.

[10] R. A. Horn and C. R. Johnson. *Matrix analysis.* Cambridge University Press, Cambridge, 1985.

[11] L. Kolev and S. Petrakieva. Assessing the stability of linear time-invariant continuous interval dynamic systems. *IEEE Trans. Autom. Control*, 50(3):393 – 397, 2005.

[12] G. Mayer. A unified approach to enclosure methods for eigenpairs. *Z. Angew. Math. Mech.*, 74(2):115–128, 1994.

[13] C. D. Meyer. *Matrix analysis and applied linear algebra*. SIAM, Philadelphia, 2000.

[14] D. Petkovski. Reduced conservatism in stability analysis of interval matrices. *J. Optim. Theory Appl.*, 70(3):597–606, 1991.

[15] J. Rohn. Stability of interval matrices: the real eigenvalue case. *IEEE Trans. Autom. Control*, 37(10):1604 –1605, 1992.

[16] J. Rohn. An algorithm for checking stability of symmetric interval matrices. *IEEE Trans. Autom. Control*, 41(1):133–136, 1996.

[17] J. Rohn. Bounds on eigenvalues of interval matrices. *ZAMM, Z. Angew. Math. Mech.*, 78(Suppl. 3):S1049–S1050, 1998.

[18] J. Rohn. A handbook of results on interval linear problems. `http://www.cs.cas.cz/rohn/handbook`, 2005.

[19] A. P. Seyranian, O. N. Kirillov, and A. A. Mailybaev. Coupling of eigenvalues of complex matrices at diabolic and exceptional points. *J. Phys. A, Math. Gen.*, 38(8):1723–1740, 2005.

[20] K. Wang, A. N. Michel, and D. Liu. Necessary and sufficient conditions for the hurwitz and schur stability of interval matrices. *IEEE Trans. Autom. Control*, 39(6):1251 –1255, 1994.

[21] Y. Xiao and R. Unbehauen. Robust hurwitz and schur stability test for interval matrices. In *Proceedings of the 39th IEEE Conference on Decision and Control*, volume 5, pages 4209–4214, Sydney, Australia, 2000.

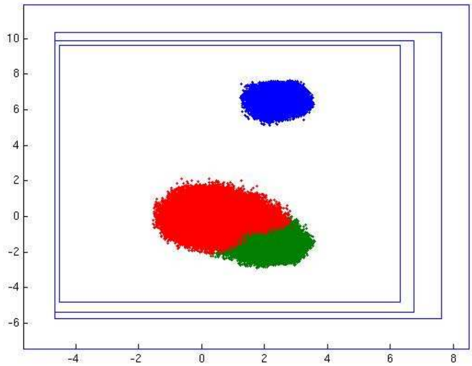

Figure 1: Eigenvalue set and enclosures to Example 1.

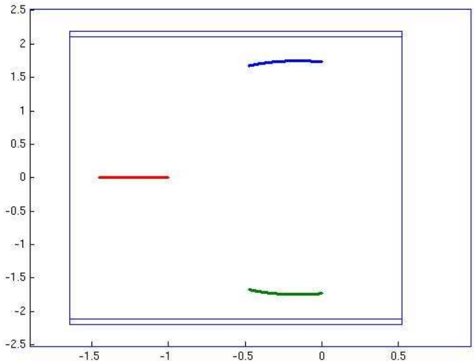

Figure 2: Eigenvalue set and enclosures to Example 2.

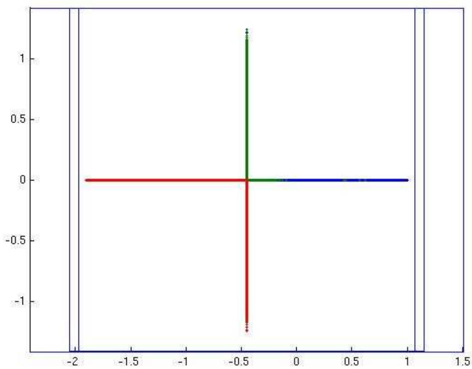

Figure 3: Eigenvalue set and enclosures to Example 3.

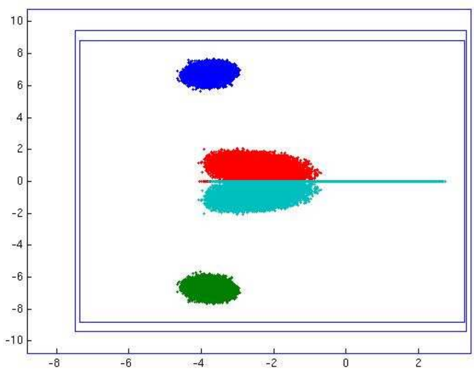

Figure 4: Eigenvalue set and enclosures to Example 4.